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ABSTRACT

This is part one of a two-part manual for teachers using SMSG high school text materials. The commentary is organized into four parts. The first part contains an introduction and a short section on estimates of class time needed to cover each chapter. The second or main part consists of a chapter-by-chapter commentary on the text. The third part is a collection of essays on topics that cannot conveniently be dealt with in the main part of the commentary in connection with a particular passage. The fourth part contains answers to Illustrative Text Items and the solutions to the problems. Chapter topcis include: introduction to formal geometry; sets, points, lines, and planes; distance and coordinate systems, angles; congruence; parallelism; and similarity. (MN)

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GEOMETRY WITH COORDINATES

PART I

SE 025 103

SCHOOL MATHEMATICS STUDY GROUP

Geometry with Coordinates

Teacher's Commentary, Part I

REVISED EDITION

Prepared under the supervision of
a Panel consisting of:

V. H. Haag	Franklin and Marshall College
Mildred Keiffer	Cincinnati Board of Education
Oscar Schaaf	South Eugene High School, Eugene, Oregon
M. A. Sobel	Montclair State College, Upper Montclair, New Jersey
Marie Wilcox	Thomas Carr Howe High School, Indianapolis, Indiana
A. B. Willcox	Amherst College

Stanford, California
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INTRODUCTION

The text that you are about to teach from is the result of collaboration between experienced high school teachers and university mathematicians. This commentary is designed to help you in several ways:

1. To help you in lesson planning.
2. To explain why we believe it is worthwhile to make certain changes from the traditional treatment.
3. To save you work by giving answers to problems in the problem sets as well as a suggested method of solution to all but the simplest problems.
4. To provide additional background information.

You can see from this list that our object in writing this commentary is to help you to present our treatment of geometry as effectively as possible.

At this point you may ask "If the text is so different and/or difficult that it needs an elaborate commentary, what is going to be the students' reaction to the text?" Our answer is as follows: Any formal treatment of geometry would be new to the student and remarks in the text commenting on why we didn't treat this or that matter in the traditional way would be meaningless to him. After all, he does not know what the traditional treatment is. Also, it is natural for us to equate being different with being difficult. It may well be that it is difficult for us because it seems so different. However, we feel certain that once you become accustomed to our terminology and treatment, you will find your geometric insight sharpened and the increased clarity obtained well worth the break from the traditional treatment. We also feel certain that clearly expressed thought in precise language cannot help but evoke real intellectual enjoyment on the student's part as well as making the subject easier for him to grasp.

The commentary is organized into four parts. The first part, in addition to the introduction you are now reading, contains a short section on estimates of class time needed to cover each chapter. At the beginning of the printed solutions for some of the problem sets, comments regarding problems in the sets are made.

The second or main part consists of a chapter by chapter commentary on the text organized as follows:

1. Remarks that apply generally to the chapter.
2. Paragraph by paragraph comments as appropriate.
3. Suggested supplementary problems that could be used either for additional assignments or possibly as test questions. These are called Illustrative Test Items.

The third part is entitled "Talks to Teachers." It is a collection of essays on topics that cannot conveniently be dealt with in the main part of the commentary in connection with a particular passage. The essays include some of the most important content of the commentary. They will be referred to hereafter in this manual simply as Talks.

The fourth part contains, first, the answers to the Illustrative Test Items and, second, the solutions to the problems.

One of our goals is the development of analytic geometry hand-in-hand with synthetic geometry, to emphasize that both are deductive systems and that it is useful to have more than one mode of attack in solving problems. We begin our development synthetically, but we are naturally led to a consideration of coordinate systems through our heavy reliance on the real number system (just as in the earlier SMSG Geometry). However, we do not make full use of coordinate systems until the student has a good command of the method of synthetic proof, acquired through considerable practice.

Obviously, we wanted to choose a development that was best from the pedagogic point of view as well as sound mathematically. Equally obvious was the fact that there was no handy formula to guide us in making such a choice. The following are some of the factors that we considered in coming to our decision.

Foremost from the mathematical point of view was the desire to choose as postulates a small number of independent statements "strategically placed" in the body of geometry so that every area of interest could be reached by a short deductive chain. At the same time, we wanted to have the postulates "intuitively reasonable". Sometimes, unfortunately, these two desires seemed to be in conflict.

In addition there was the choice of technical language. On the one hand it was desirable to use as technical words those that have strong connection with colloquial usage and our intuitive geometric ideas. On the other hand we did not want to have our mathematical language so close to colloquial usage that there was danger that the wrong one of several possible meanings might be taken. In the interest of good teaching of mathematics we can hardly overemphasize the pedagogical importance of reinforcing for your students the strict meaning of terms as given in the text by consistently using them only in their technical sense.

In connection with the aim to provide additional background information there are several things to be said. First, it is obvious that, in a textbook at this level, many discussions have to be logically incomplete. We have cut some corners, expecting the students' intuition to take over, and we believe this is as it should be. Often, in fact, it would take considerable argument with most students to convince them that they had indeed depended on their intuition. On the other hand, some students may see this dependence and ask how the argument may be revised to avoid such a question. The running commentary is designed to help you when this happens. As a further aid you should have available a copy of "Studies in Mathematics," Volume II, Euclidean Geometry Based on Ruler and Protractor Axioms, by C. W. Curtis, P. H. Daus, and R. J. Walker. It contains, expecially in the first chapters, much material that could have been put in the Talks. It also contains detailed proofs of basic theorems that are not mentioned in the text. The properties stated in these theorems are intuitively obvious and are generally accepted by students without comment. A completely logical development of geometry must, nevertheless, contain proofs of these theorems, and so they are included here for whatever use you wish to make of them. When we refer to the reference volume, we will speak of it as "Studies II." Inasmuch as "Studies II" was specifically written for the previous SMSG Geometry text, the G edition, as opposed to this which is the GW edition, there is a short essay in the Talks establishing the necessary connection between the terminology used in "Studies II," and ours.

Some teachers may enjoy referring to a lighter presentation of some geometric idea. To them we suggest "Studies in Mathematics," Volume V, Concepts of Informal Geometry. Information about the books can be obtained from School Mathematics Study Group, Cedar Hall, Stanford University, Stanford, California.

Although we felt it unwise to make our text logically complete in its theorems and proofs, we did attempt to give a complete foundation of postulates and definitions. On such a foundation a student can build as elaborate and complete a structure as his capabilities permit, with the help of his teacher and of supplementary reading.

Obviously you will like some features of this text better than others. In any case, we hope that you, will teach it following the presentation of the text and taking into consideration priorities and emphases suggested in this Commentary. Suggestions for improving the text are invited; send them to School Mathematics Study Group, Cedar Hall, Stanford University, Stanford, California.

A WORD ABOUT PROBLEM SETS

The fundamental purposes of the problem sets are:

1. To reinforce the students' comprehension and appreciation of the ideas and techniques being developed;
2. To develop the students' ability to recall accurately and fully concepts once understood;
3. To enhance the students' power to apply combinations of concepts and techniques effectively to the solution of problems;
4. To challenge the students to analyze problems and to probe for solutions not readily apparent; and
5. Occasionally, to orient the students' thinking toward a concept not yet introduced but to be developed in the succeeding text material.

As a general rule, each problem set will commence with a number of simple exercises designed to "drill home" the basic concepts. After these, the difficulty of the problems will increase, roughly with their order of appearance, the last one or two, as a general rule, being challenge-type problems.

It is hardly practicable, however desirable, for every problem set to be, or to contain identifiably, a "proper" homework assignment for a given student. We have attempted to provide "good" problems, and enough of them to allow the teacher some freedom of choice. Teachers are urged to read the problems of each set, and possibly their solutions, with a view to tailoring homework assignments to fit situations.

There may be certain problems in a set, perhaps exploratory in nature and leading into succeeding sections, which the authors strongly recommend assigning to every student. Such problems are "starred," the asterisk meaning "Do not omit."

With regard to "theorems" in the problem sets, there will probably arise some questions about their status. Students may want to cite the solution of one proof-problem as a basis for a deduction in solving another problem, or indeed in proving a subsequent theorem not in a problem set. The teacher may wish to prescribe ground rules for this, and we would suggest that a "safe" rule would be not to allow it at all. However, there may be occasions for relaxing such a rule.

One final remark about proofs: The partial "proofs" which appear as completion-type problems in the problem sets should not be considered as models of "correct" proofs in a restrictive way. We intend them to illustrate acceptable proofs, but not to be the only acceptable ones. Good, original reasoning can be presented in many forms, all orderly and convincing, differing primarily in their respective appeals to personal taste.

Following the general commentary section for each chapter (except Chapter 1) problems suitable for use in a chapter test have been listed. It should be clearly recognized that the list is often too long and may contain too many similar items to constitute a single chapter test, but it was designed to provide samples of problems which a teacher might use in assembling such a test. A complete chapter test might be constructed by selecting a representative combination of these test items. In most cases enough problems testing the same points have been included so that a teacher with several class sections could make different but similar tests.

USING THE TIME AVAILABLE

This text was written so that very good classes will have ample material to challenge them for a year. It follows then that most classes will not be able to cover the entire material. You may prefer not to rush through important topics just to cover pages, so this note will suggest some of the possible choices that you can make. For example the time schedule below suggests two approaches, one for the average class and one including vectors, for the advanced class.

A full course gives adequate coverage of all the exposition and a substantial number of problems from each set. There is an abundance of problems because this was felt to be necessary to meet the needs of all the students. Certainly there are more here than any student could be expected to work. Also there are some problems that only a few of the students are expected to solve.

The time schedule given below is the result of the opinions of the secondary teachers who participated in this writing project and of the experience of teachers who used the preliminary edition. The time allotment serves to indicate the relative amounts of time for the various chapters. Whether your class be average, above average, or definitely superior the relative amounts of time devoted to each chapter as indicated in the chart should give a well-balanced course. If you find that these relative times are not working out in your classes, we suggest that you consider carefully the depth to which you are teaching the various chapters. The remarks in this commentary were designed to be as helpful as possible on this question of depth.

Chapter	No. of Days	Cum. Tot.
Part I		
1. Introduction	2	2
2. Sets, Points, Lines, and Planes	8	10
3. Measurement of Distance and Coordinates on a Line	16	26
4. Angles	13	39
5. Congruence	21	60
Part II		
6. Parallelism	13	73
7. Similarity	13	86
End of First Semester		
8. Coordinates in a Plane	30	30
9. Perpendicularity, Parallelism, and Coordinates in Space	16	46
Part III		
10. Vectors	*	*
11. Polygons and Polyhedrons	16	62
12. Circles and Spheres	25	87
Total for year		173

*In classes of superior students where it is desirable to study vectors we anticipate that the time required is about ten days. With such students it may be possible to shorten slightly some of the times listed above, particularly the chapters on coordinate geometry and congruence. There are some theorems in Chapter 8 that are treated in the chapter on vectors. These may be omitted from the class presentation of Chapter 8 if vectors are presented. Details appear in the Commentary on Chapter 8.

Let us stress again that we do not feel we can set a rigid time schedule. Yet we feel reasonably secure in saying that you should not spend more than thirteen weeks on Part I. You will not accomplish this, even with an advanced class, if you linger too long on Chapters 3 and 4. With an average class you will want to approach many of the proofs from an intuitive basis or by giving illustrations using specific situations. In some cases you may wish to skip the proof entirely after discussing what the theorem says. All this is treated more fully in Chapter 3 and 4 of the Commentary.

Neither will you be able to meet your schedule if you start your students doing proofs too early. Several times we cite Chapter 5 as the time for students to venture out alone. This is about the eight or ninth week. While we presented numerous proofs in the interest of having a complete geometrical framework, the student is not expected to be able to produce them at this stage of the game. Spending time at this task at this time would possibly prohibit your considering other topics which are a vital part of the course. Chapter 5 deals specifically with techniques of proof. This is the spot where students should first consider the problem of constructing their own proofs.

We are working on the theory that it is good to have an excess of food on the table. The wise person does not try to eat everything before him, but judges his capacity, evaluates his wants, and acts accordingly. So it is in geometry, but here our wants differ greatly, being affected by time, teacher preferences, local syllabi, and individual and group differences. We recognize that numerous plans are possible to meet the limitations imposed by these considerations. However, if you are trying to decide what food to choose when you judge that your capacity will not permit you to make the full round of the table, we suggest the following as what can be omitted, in the order very roughly, in preference of omission, the last item being the one you should least consider omitting.

1. Vectors
2. The proofs of Chapter 9. (Treat the chapter entirely on an intuitive basis.)
3. The sections on polyhedral angles and polyhedrons in Chapter 11.
4. The detailed proofs in Chapters 3 and 4. (Treat the chapters on discussion and problem solving basis.)
5. The sections on the coordinate geometry of the sphere in Chapter 12.
6. The sections of Chapter 9 on the coordinate geometry of lines and planes in three dimensions.

Finally, for the class, or the individual student, needing more material than all the numbered chapters provide, the appendices may provide the basis for a number of interesting assignments.

Chapter 1

INTRODUCTION TO FORMAL GEOMETRY

In this chapter we discuss informally what formal geometry is and the distinctions between it and physical geometry. Our immediate goal is to give the student some idea of the format of a postulational development of geometry and to prepare him intellectually so that he will be willing to work with

1. undefined terms
2. postulates
3. chains of deductive reasoning.

Ideally, it should be possible to start the text with a list of undefined terms on page 1, followed by postulates, then theorems and proofs. We all know, however, how difficult it is to play intelligently or even be an interested spectator of a game in which we have no idea of the rules or how to keep score. In a sense we are trying in this chapter to tell the students who the players are, what the rules are, and how we keep score. We hope that most students will become active, intellectual participants.

If you have never seen or played a game of cricket but have tried by reading a description of cricket in an encyclopedia to "understand" the game, you probably have some idea of the student's problem in trying to "understand" what formal geometry is about, if he has never worked with it. An effective way of learning about cricket would probably be to read the encyclopedia article to get a general idea of the game and then to watch a game in company with a friend familiar with it. We strongly suggest that the use of this chapter should be similar to that of the encyclopedia article. As soon as he gets a fair idea of the "game," he should go on with his teacher as a guide and start participating in it. Occasionally, it may be appropriate for the teacher to refer him back to this chapter.

Needless to say, we are well aware that many of the philosophical matters we touch on in this chapter are subjects to which entire treatises are devoted. We are also aware that the student at this stage of his development should not be asked to pursue these topics to any depth. We do suggest, however, that additional material in Talks, entitled, "Facts and Theories," may be helpful to the teacher. There is also a list of books at the end of Chapter 1 in the text from which a more detailed history of the evolution of geometry may be obtained as well as biographical sketches of outstanding contributors to geometry.

For students with a strong background in formal mathematics, this chapter will serve largely to remind them of the general structure of formal mathematics. Since this is then largely in the way of review, at least to the level needed to start Chapter 2, these students need spend little time on Chapter 1. For students with weak backgrounds in formal mathematics, for whom the discussion is largely new, there is a limit to what can be gained by talking about the axiomatic or postulational development of mathematics without working with a specific axiomatic development. Hence, even if their understanding is poor, if they are willing to "play the game," they should go on to Chapter 2. Thus, for any type of student, we conclude the time spent on this chapter should be brief without excessive effort to examine the ideas presented in their full depth.

In summary of our general remarks we reiterate that, as soon as the students have a fair idea of the "game," involving undefined terms, postulates, and deductive reasoning, they should begin Chapter 2.

The numbers in the left margin refer to the pages in the text to which the comments relate. At the top of the page are the page numbers of the text that the commentary refers to and, also, the numbers of the sections.

1 Presumably at the first meeting the students have not had an opportunity to read the text. If so, the question "What is geometry?" might be a useful way to lead them into the Introduction. This query elicits a great variety of answers. These will undoubtedly give you the opportunity to begin to draw the distinction between the informal geometry that they have had up to now and the formal geometry they are about to start.

The students will be quick to recognize that they have acquired a number of geometric facts, informally, in previous years. They need to realize that now they are about to use a formal approach, one that is deductive in nature. They are about to begin proving formally what they have already been using and accepting as true.

The transition from informal acceptance of facts or acceptance based on intuition and/or induction to the formal approach is not always easy. The student will encounter some facts early in the year that are so simple or obvious to him that he wonders if they are worth putting into words. Included in Chapter 1 are some examples to show that the deductive method can be used to arrive at results that are not easy to see intuitively.

The student needs to realize (and this realization does not come quickly) that in geometry we take a few simple statements and from them deduce more complicated and less obvious relations. Furthermore, these deductions along with the definitions and postulates can be listed in a way so that any deduction is based on the statements that precede it.

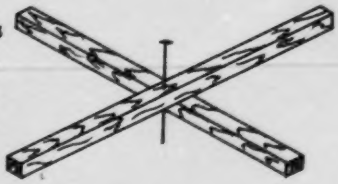
As we attempt to display this organization we face compromises because that which is mathematically needed is not always pedagogically feasible. When we felt it necessary to compromise and leave a gap we have tried to acknowledge it and, where practical, supply the missing mathematics in the Appendices, the Commentary, or in Studies II.

4 The Exploratory Problem is discussed in the text on page 6 and, finally, on pages 12 and 13, Conjecture (X) is established. Also see Problem 9, Problem Set 1-4, on page 9.

The emphasis of this section on the limitations of the inductive process might lead the student to think that induction is not useful. The student should be aware that induction is useful in many situations; in fact, there are cases when the scientists find it the only tool available.

The statistician is well aware of the inductive nature of his conclusions and is always careful to qualify his assertions appropriately. For instance, after investigating a small portion of a given population, he may make an assertion about the whole population qualified by a remark concerning the reliability of his conclusion.

An example of the use of inductive reasoning that probably would appeal mostly to the boys in the class concerns police methods. Many cities keep files listing the modus operandi or style of operating of various criminals. If a crime is committed in a certain manner, the detectives refer to these files to identify criminals who habitually operate in this manner. The police, reasoning inductively, consider the criminals thus identified to be well worth investigating.



6 An easily constructed device for illustrating Problem 1 consists of two short strips of balsa-wood or soda straws pierced by a pin in the center of each. This device permits the demonstration of a number of cases quite rapidly. A more elaborate presentation can be obtained by running the pin through the center of a circular protractor so that the angles can be immediately determined.

It must be emphasized that this device, as well as other such devices you may find convenient to use, should be clearly identified in the student's mind as objects belonging to physical geometry rather than formal geometry. Manipulations with a physical device can never constitute a formal proof.

In spite of these limitations, teaching aids such as these, are an excellent means of starting the student on the way to understanding the ideas involved. There will be cases where this understanding can be enhanced if the student makes the device himself or perhaps invents one to illustrate some principle. Often physical models or devices offer the best and quickest way of presenting an idea.

10 A student who wonders what an infinite set is may be satisfied by the statement that a set is infinite if and only if it has at least as many elements as there are positive integers. Of course, this answer may raise the question as to the meaning of the phrase, "at least as many elements as there are positive integers." This means that there is a one-to-one pairing between the set of positive integers and some subset of the given set. The difficulty with this more precise form of the answer is that the idea of a one-to-one pairing may not be familiar to the student at this stage.

Another way of answering the student is to say that a set is finite if and only if "it can be counted." Of course, this answer may raise the question of what it means to count a set of objects. To count a set of objects means to establish a one-to-one correspondence between the objects of the set to be counted and a subset of consecutive positive integers of the form $\{1,2,\ldots,N\}$ for some positive integer N. This explanation may be difficult for the student who has not thought about one-to-one pairing and its connection with counting. Also it is a little removed from his question about infinite sets, since a set is infinite if and only if it is not finite. Further, to be technically correct in this way of answering the student, we must agree that although the empty set cannot be counted, it is finite. After one-to-one pairing and the empty set have been discussed, it may be easier to talk with students about infinite sets.

10-12 We first comment generally on pages 10-12 of the text, then return to specific comments related to the example on page 10.

The description in the text of what is involved in setting up a mathematical theory emphasizes the role of postulates and deductions. (The role of definitions and undefined terms is discussed in more detail in the next section.)

It took the human race a long time to develop the idea of a mathematical theory. You cannot expect your students to grasp it from an abstract description. The understanding of what is involved in logical reasoning will grow throughout the course as students actively engage in logical reasoning. Nobody can learn logical reasoning in a vacuum.

Only a very remarkable student will fully understand the paragraphs about theorems, postulates, proofs, and undefined terms, when he first studies this chapter. These ideas will come into sharp focus in the student's mind only when he has had some experience with them.

As you consider the meaning and significance of the postulates it may be useful to note that:

1. Until about 1800, everybody believed that the postulates of geometry were "self-evident truths," and that the theorems proved from them were statements of fact about the outside world, learned by pure reason.

2. Since the discovery of non-Euclidean geometry, it has been plain that the postulates of ordinary geometry are not "self-evident truths." There are many kinds of

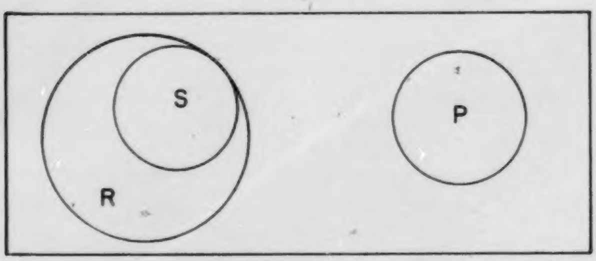
geometry that are equally valid, logically; some of the very "peculiar" ones are useful in physics; and each of them is described by its own set of postulates. Postulates, therefore, are simply descriptions of the kind of geometrical theory that we propose to investigate at a given time. And when we prove a theorem, we are not showing that the theorem is "true" in the sense that it fits the facts of the outside world. When we prove a theorem, we are merely showing that the theorem holds true in the mathematical system described by our postulates. (See the remarks on non-Euclidean geometry in the chapter on parallels, and the Talk on Non-Euclidean Geometry.)

However, it does not seem to us that this second viewpoint in all its detail is suitable for presentation in the first week. The student would probably be completely bewildered, and he might get the idea that Euclidean geometry is just words, words, words. We have, therefore, been treading a rather fine line, explaining to the student approximately as much as we think that he can understand, and being careful in the process not to make any statements that will have to be corrected later.

What needs to be emphasized, at the start, is that postulates are not just pulled out of the air to satisfy somebody's whim. The space of Euclidean geometry is an extremely good approximation to physical space. This is why it got invented, and this is the most effective way to think about it. We can and we should use our intuition of physical space to help us guess what can be proved and how we can prove it. The proof itself, when we get it, has to be based logically on the postulates. A mathematical system, like the geometry we are developing, which consists of postulates and theorems involving undefined and defined terms, is called a deductive theory. This theory itself is given meaning and content by exhibiting an interpretation of the undefined terms. When we give the usual interpretation of point, line, and plane from physical space we get our physical geometry, which is an approximate model of our deductive theory. Other interpretations of the undefined terms lead to different models. A further discussion of mathematical models and how they work is given in the Talks.

It might be well to return to this chapter after the student has had a fair amount of experience with the concepts which we have been trying to explain. After the class has finished Chapter 5, the ideas of postulate, theorem, proof, and undefined terms should have become entirely comprehensible.

10 The discussion of H_1 and H_2 leading to the conclusion that no square is a pentagon could be expressed in terms of Venn diagrams, if the class is familiar with them. If we let the universal set be the set of geometric figures, then the first statement, H_1, means that the set S of all squares is a subset of the set of all rectangles R. The second statement, H_2, means that the intersection of the set of all pentagons P with R is the empty set.



From this figure, the student may find it easier to understand the deduction: No square is a pentagon. He can see readily that the intersection of S and P is empty.

When Section 1-7 is studied the teacher may find it helpful to use Venn diagrams to illustrate the meaning of special words. For example, to make the distinction between the use of "some" and "all," we might make some contrasting statements involving the imaginary objects Satis, Ratis, and Patis. It would be equally convenient to use any other nonsense names such as Sip, Rip, and Pip, or some other words whose initial letters are S, R, and P . We use imaginary objects since some of the statements we wish to make would not be true if they involved squares, rectangles, and pentagons. Suppose that we consider first the argument:

H_1: All Satis are Ratis;

H_2: No Rati is a Pati;

Conclusion: No Sati is a Pati.

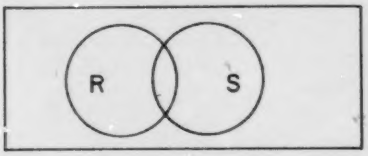
If we denote the set of all Satis by S , the set of all Ratis by R , and the set of all Patis by P , then we see that a Venn diagram interpretation of the relation among our statements is exactly the same as the one above for the argument concerning squares, rectangles, and pentagons.

Now consider the statements

K_1: Some Satis are Ratis;

K_2: No Rati is a Pati.

Since "some" admits the possibility of "not all," a Venn diagram interpretation for K_1 might be:



Note the difference between this possible configuration for R and S , and that in the earlier diagram. This difference is due to the difference between "all" and "some." To continue by considering K_2 , we see that the set P must lie outside R (this is the significance of "no"), but that it may or may not have elements in common with S . The conclusion we can deduce in this case, in contrast to our earlier conclusion, is that "some Satis are not Patis."

Clearly, this discussion does not exhaust the possibilities of effective use of Venn diagrams in getting students to visualize set relationships.

16-18 The student should come to realize by the time Chapter 2 is completed, that it is useful and permissible to define other words in terms of the undefined terms. Definitions of this sort are clearly abbreviations for longer phrases involving the undefined terms.

He also should come to realize as the course progresses that the postulates "define" in a very effective way the terms point, line, and plane.

The question of definition vs. undefined terms is discussed more fully in Section 2-4.

This chapter is not followed by a list of Illustrative Test Items as is the practice in the later chapters. We do not feel that it is appropriate to test directly on the material of this chapter at this stage. Our aim has been to get the student ready to begin formal geometry. The best test of success in attaining this goal is to observe the students' progress in Chapter 2.

Chapter 2

SETS, POINTS, LINES, AND PLANES

From the teaching standpoint this chapter has two main divisions:

I. An informal introduction to set theory in Sections 2-1 and 2-2 and a presentation of the notion of one-to-one correspondence in Section 2-3.

II. The start of the formal axiomatic development of geometry with the incidence postulates in Sections 2-4, 2-5, and 2-6, and 2-7.

These divisions present somewhat different teaching problems.

The discussion of sets as we have presented it, is not really a mathematical theory, but simply an explanation of the language in which we propose to talk. As the examples show, all of the basic ideas about sets--with the sole exception perhaps of the empty set-are already familiar as they occur in specific examples. Only some of the words and the consequent abstractions in which we talk about these ideas are new. As soon as you feel satisfied that the students feel moderately at home with the technical terms--intersect, intersection, empty set, union--you should move on to the rest of the chapter. These terms will be used repeatedly throughout the remainder of the text and thus, there will be ample opportunity later to reinforce and strengthen their appreciation of these terms.

For some students the sections on sets will be in the nature of review. They may in addition be familiar with the standard notation of set theory. For these students we have provided problems using this notation as well as an appendix, Appendix I, entitled A Convenient Shorthand. This material is intended to be strictly optional, and the title of the appendix is meant to suggest the spirit in which the notation is to be regarded. There is a serious danger in talking too much, or in being too sophisticated about sets, at the high school level: the impression may be conveyed that writings like $(A \cup B) \subset C$ is a loftier occupation than proving meaty theorems and solving hard problems in geometry and algebra. This would be sad. We therefore, believe that the language of sets should be introduced matter-of-factly without fanfare, and that the notation of a set theory should be taught to a given student when, and only when, the student is prepared to think of it as a matter of convenience. However, the language of sets is going to be used continually. For example, an angle will be defined as the union of two noncollinear rays. As another example, two lines in the same plane are parallel if they do not intersect; this means that the lines, considered as sets of points, have no member in common.

The notion of one-to-one correspondence is a familiar idea for most students. It is used actively in connection with coordinate systems on a line in Chapter 3, with ray coordinates in Chapter 4, with the notion of congruence in Chapter 5, and generally throughout the remainder of the text.

We recommend that as soon as the students feel moderately at home with the technical terms--intersect, intersection, empty set, union--and have the idea of a one-to-one correspondence, you should move on to the formal development of geometry.

The chief teaching problems, as we see them, related to the material in the second division in which we start formal geometry are:

1. The problem of getting the student to distinguish between his informal geometric ideas and the formal concepts as introduced by postulates and through definitions.
2. The problem of getting the student to understand exactly how much, and how little the postulates assert, and to realize that when we define a technical term or state a postulate, we mean what we say, and say what we mean.

3. The problem of helping a student to understand a deductive proof.

We have attempted to head the student in the right direction with respect to the problem above by our remarks in Section 2-4 which in turn reinforce some statements in Chapter 1. It is, however, next to impossible to do a complete job without exposing the student to the postulational development itself. Hence, we suggest a brief treatment of Section 2-4 with appropriate reinforcement of the ideas presented there as the student works with the material in Sections 2-5 and 2-6.

The teaching problems stated as (2) and (3) above are closely intertwined. Our recommendation is that the main emphasis in both Sections 2-5 and 2-6 be on what the postulates say. For instance, in discussing a problem, such as Problem 4 in Problem Set 2-5, instead of asking:

> "Is the justification under discussion really a proof?", and thus, emphasize the concept of proof: ask instead, "Do the postulates really permit us to say so-and-so?", thus, emphasizing how much and how little the postulates say.

This procedure also helps the student to understand the proof. This focusing of attention on the postulates also makes it easier to meet the question sometimes raised on the need to prove the "obvious." In addition, we do not feel that much can be expected from students at this stage, in the way of proofs, although we want them to get started on them. We feel that we should not expect proofs from all of the students until we reach Chapter 5.

19 The omission of the number that is required for a baseball team was intentional. We tried by this omission to indicate to the students that it is a set whether you know the number of elements or not. The number of elements in a set is a property of the set,but it is not essential to the fundamental notion involved.

Since the mention of a team as a set brings to mind definite relationships that exist among the members of the set, the example involving the Empire State Building is used to dispel the idea that such relationships have to exist before the objects can be considered to be members of the same set. Of course, after we place them in a set we establish at least one relationship between them, namely, that they belong to the same set.

20 From time to time questions of the completion type will be asked with the request to "Fill in the blanks." If the books are to be used again next year you may want to instruct the students to "fill in the blanks mentally" or "Write the answers on another sheet of paper to be turned in or considered in class."

22 We apparently use the word contains in two ways. However, when we say that "the set $\{3, 6, 7, 9\}$ contains 6" we are actually presenting a version of the sentence "the set $\{3, 6, 7, 9\}$ contains the set $\{6\}$." Using the convenient shorthand discussed in Appendix I, "contain" corresponds to "$\supset$" and we consider that $A \subset B$ (A lies in B) expresses the same ideas as $B \supset A$ (B contains A). (This is analogous to the correspondence between the expressions $3 < 5$ and $5 > 3$). Note that we may write in symbols $6 \in \{3, 6, 7, 8\}$ but may not write $6 \subset \{3,6,7,9\}$ because $\subset$ (and thus $\supset$) expresses a relation between sets. Instead of incorrect use of symbols $6 \subset \{3, 6, 7, 9\}$ we have our choice between $\{6\} \subset \{3, 6, 7, 9\}$, $\{3, 6, 7, 9\} \supset \{6\}$, or $6 \in \{3, 6, 7, 9\}$. The fact that there is a real distinction to be made between 6, and $\{6\}$, (that is between the element 6 and the set whose only element is 6) is discussed on page 23 of the text. As far as the student is concerned, this discussion in the text should suffice to make the possibility of a distinction clear and there is no need to labor fine points of language at this stage.

24 We make a distinction in our use of the words "intersection" and "intersect." This distinction is presented to the student on page 27. We comment on it here so that you will be forewarned and can look ahead now.

Some classes may have some background in set theory. If so, it is possible that our use of "intersection" and "intersect" is at variance to the usage to which they are accustomed.

In set theory any two sets, A and B, "intersect" though they may have no points in common. However, for our use in geometry, the definitions as expressed here, are more suitable. We might summarize the situation in this manner:

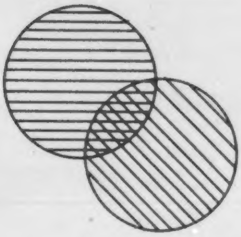
The phrase, "A intersects B" means that A and B have at least one point in common, while an expression involving the noun, intersection, such as "the intersection of A and B," describes what may or may not be an empty set.

24 In the first figure, the arcs were not completed as circles; we wished to avoid suggesting to the student that we are concerned with the overlapping areas.

The intersection would still be {P, Q} if the arcs were extended to form circles, because we distinguish between a circle and the region enclosed by the circle. This distinction is considered later in the text. However, should you wish to consider this distinction now the following paragraphs are pertinent.

When we say circle, we mean the set of points that make up the circumference. Mathematicians have come to call the set which is the union of the circle and the points that it encloses the disc or circular region to distinguish this from the circle itself.

If you consider, instead of the intersection of two coplanar circles, the intersection of the corresponding two discs or circular regions, you would have five cases. (1) If the circles do not intersect and each is outside the other, the intersection of the discs is the empty set. (2) If the circles are externally tangent, the the intersection of the discs is a point. (3) If the circles intersect in the manner described in the text, the intersection of the discs consists of those points that make up a region that is shaped like a convex lens as indicated by the cross-hatching in the diagram. (4) If one circle is inside the other or is internally tangent to it, the discs intersect in the smaller circular region. (5) If each circle contains the other, the discs intersect in the circular region itself.



The intersection of the two lines is a set and is therefore properly denoted as $\{W\}$.

26 The student might object to calling something devoid of elements a set. The idea of the intersection of two sets being an empty set, which we call $\emptyset$, is logical, but it seems strange to think of this, the empty set, as being a subset of a set A. This can be justified. For by definition of subset, if $\emptyset$ were not a subset of A then there would be an element that is an element of $\emptyset$ but not of A. But, this is not the case, since $\emptyset$ is empty. Hence, the empty set is a subset of A. Indeed, since A could be any set, we have shown that $\emptyset$ is contained in every set.

You may notice that we speak of the empty set rather than an empty set. The justification for this is not always easy for students to grasp and we have not tried to justify it for them, nor do we suggest that you try to. For your information we will, however, give a justification. Recall first that two sets are considered to be the same if and only if each is contained in the other. Suppose that there were two empty sets, $\emptyset_1$ and $\emptyset_2$. We shall prove that $\emptyset_1$ and $\emptyset_2$ are the same. First, since $\emptyset_1$ is an empty set, it is contained in every set. But $\emptyset_2$ is a set, hence, $\emptyset_1$ is contained in $\emptyset_2$. Similarly, since $\emptyset_2$ is an empty set, it is contained in every set. But $\emptyset_1$ is a set, hence, $\emptyset_2$ is contained in $\emptyset_1$. Thus, we have shown both that $\emptyset_1$ is contained in $\emptyset_2$ and that $\emptyset_2$ is contained in $\emptyset_1$.

Consequently, there is only one empty set. Hence, we speak of *the* empty set rather than *an* empty set.

The teacher who wishes further information on sets is referred to E. J. Mc Shane, "Operating with Sets," in the *Twenty-Third Yearbook* of the National Council of Teachers of Mathematics.

26 Sometimes students confuse the empty set, or $\emptyset$, with 0. There are two aspects to be considered:

(1) How to prevent such confusion arising in the first place.

(2) How to dispel the confusion if it has unfortunately arisen.

In the first place we try to avoid such confusion by not placing 0 and $\emptyset$ in close juxtaposition either in our writing or in our speech. For instance, we do *not* say to the student that we use $\emptyset$ rather than 0 to denote the empty set in order to distinguish the empty set from zero. Again, we do *not* say the empty set is a set consisting of zero elements. To avoid the juxtaposition of empty set and zero, we say that the empty set is a set consisting of *no* elements at all.

Suppose that, in spite of our care, confusion between the empty set, $\emptyset$, and 0 has arisen. The example involving $x + 1 = x - 1$ at the end of the paragraph on page 26 may be helpful. (Note that the text does not state that zero is not a solution. This is in keeping with (1), above.) We have seen that the set of numbers satisfying this equation is the empty set. If 0 were the empty set then this last sentence would read: We have seen that the set of numbers satisfying this equation is zero. This sentence is nonsense, because zero is not a set, but even if a student cannot see this reason for its being nonsense, he can certainly see that zero is not a solution of $x + 1 = x - 1$. A similar illustration may be built up using the solution set over the *real* numbers for the equation $x^2 + 1 = 0$. As another illustration note that 0 is a solution of $3x = 0$, while the solution set is *not* empty. As still another illustration one may ask a student if there is a difference between being given a zero after an absence from a class exercise and having no grade recorded in these circumstances.

30 Throughout the section on one-to-one correspondence there is a clear implication that two non-empty sets have the same number of elements if and only if there is a one-to-one correspondence between them. It is interesting to note that a child who cannot count could determine that a basket of apples and a basket of oranges each had the same number of objects in them by the simple expedient of "pairing" the apples and oranges. This is one reason why the mathematician considers one-to-one correspondences to be more fundamental than counting and why he chooses to define the idea of "same number" in terms of one-to-one correspondences.

In studying this notion of "same number" the mathematician notes that some sets have the remarkable property that they have the same number of elements as some proper subset of themselves (a proper subset of a set is a subset that is not the whole set). For instance, the set of positive integers has a proper subset, the set of positive even integers. These two sets can be paired in a one-to-one manner as follows: $1 \longleftrightarrow 2$, $2 \longleftrightarrow 4$, $3 \longleftrightarrow 6$, ... and in general n is paired with $2n$. A set with this remarkable property is said to be infinite. Among the sets that are infinite is the set of all real numbers. In Chapter 3, we postulate that there is a one-to-one correspondence between the set of all real numbers and the set of all points on a line. As a consequence of that postulate we can then say that a line has infinitely many points on it.

A key use of the one-to-one correspondence concept occurs in the text in connection with our treatment of congruences between triangles. We use it to make explicit in our notation which parts of one triangle are paired with parts of the other.

36 In starting the formal part of geometry, may we remind you that it may be convenient to refer to, or refer the student to, the Section 1-7 on Special Words and Phrases in Chapter 1 of the Text.

In past years many texts have made distinctions between "axiom" and "postulate" following the pattern used by Euclid. Generally an axiom had been considered to be a "self-evident truth" or a "common notion," to paraphrase Euclid. Thus, it was a general statement as opposed to a postulate, which was described as a "geometrical axiom."

Modern usage, due to our deeper understanding of the relations between fact and theory, does not make such a distinction. In our text, the words "axiom" and "postulate" are used interchangeably, though the latter occurs more frequently. For us the "postulational approach" is the same as the "axiomatic approach."

Our postulates and definitions use words, such as "any," "contains," "exactly," "distinct," and other words that may have been discussed in Chapter 1, but not formally defined. You may find words that have not been mentioned before. The awareness of our desire for preciseness in language may prompt the dutiful student to challenge this seeming laxity.

He could be reminded that in a given set of postulates and definitions for developing a geometry it is hardly to be expected that the laws of classical logic, the rules of grammar, and a definition of all the terms can be included. We recognize their need and assume them whenever needed just as we assume familiarity with set theory and with the usual laws of arithmetic and algebra. However, there may be times when in spite of our care a definition or other logical assumption is over-looked, because we are so intent upon the particular geometric topic under immediate consideration.

Postulates are numbered consecutively throughout the text. Theorems are numbered consecutively within each Chapter. In the back of the text, before the index, there is a list of the Postulates and Theorems. Definitions are not numbered or listed in the back of the text. They may be found by using the index.

37 We have, by no means, started off defining a line despite the fact that Postulate 2 starts off in the same manner as many definitions, "Every line is ...". One of the things that can be pointed out about a definition is that it can be "reversed," that is, that it amounts to an "if and only if" statement. This postulate certainly fails by this test to qualify as a definition when we consider the result obtained if we attempt "reversing" it.

Because the introductory geometry that most teachers are familiar with accompanies "line" with the word "straight," there may be some reluctance to give up this adjective. Please note that our postulates which relate to lines are designed so that they describe only "straight lines." This being the case, the adjective "straight" is superfluous. (See also the general discussion on the separation postulates in Chapter 4 of this Commentary.)

The remark, that we believe our development is to be more instructive since we do not assume many points, is intended quite literally. If we had assumed many points in our postulates, much of our later discussion such as that leading to Theorem 2-2 and 2-7 would not be necessary and at the same time we would have lost some easy illustrations of how our postulates interact with each other.

37-38 As the student studies his first proof, we should keep in mind that our aim is to help him to understand the proof. This does not mean a rote memorization of it. Remember for the student at this stage, the role of the postulates and definitions is more important than the formal aspect of proof itself. In noting the role of postulates and definitions the student should be led to realize that what counts in our formal work is what is actually stated in the postulates and definitions, not what led us to state them or what might have been stated. He has to become somewhat of a "lawyer" in his use of them.

There will be ample time later to develop the techniques of proof. Teachers have their own ways of doing this: Changing the labels on the figures, encouraging students to come up with different proofs, challenging the exceptional student to disprove something, changing from a paragraph form to the two-column form and vice-versa. We must aim to encourage the student in creative mental effort and not to emphasize recitation based on memorization.

43 In discussing the proof of Theorem 2-5, point out to the students the usefulness of a sketch to help keep track of the development of ideas in the proof as well as the notation. A suggestion to this effect appears in the text following the statement of Theorem 2-6.

47 Postulate 8 fills the blank appropriately in the proof of Theorem 2-8. The hypothesis that the plane does not contain the line is thereby contradicted, and this proves the theorem since we know the plane and line have at least one point in common (since they intersect) and have shown that the intersection cannot contain more than one point.

47-48 Postulates 2, 3, 6, 8, in that order, fill the blanks appropriately in the proof of Theorem 2-9. There is no other plane containing $\ell = \overleftrightarrow{AB}$, and P. Thus, we have shown not only that there is a plane which contains the point and the line but that it is the only plane which contains them.

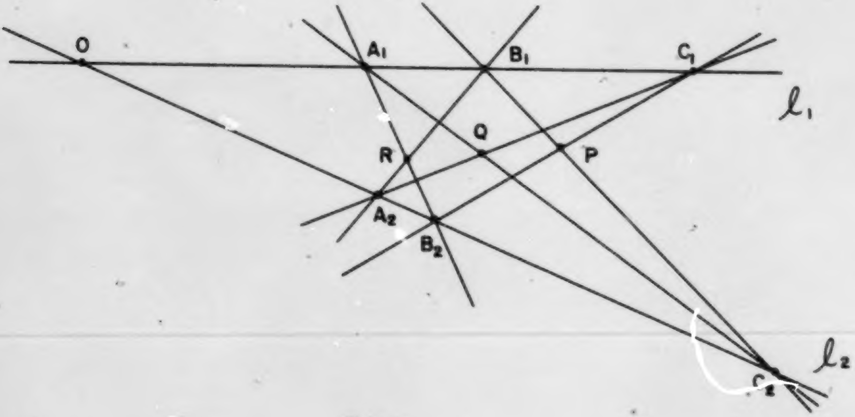
48 In Theorem 2-10, appropriate fill-in words are Theorem 2-4, Postulate 2, Postulate 2, Theorem 2-9 (just proved), and Postulate 8, in that order. The existence of a plane containing ℓ , was assured by the use of Theorem 2-9, the fact that the plane, which contains both P and B of ℓ_2, contains ℓ_2 was assured by Postulate 8. Since the plane obtained by the use of Theorem 2-9 was the unique plane containing B and ℓ, there is only one plane containing ℓ_1 and ℓ_2. This proves the Theorem.

There are some possibilities for enriching the material in this chapter that might be suitable for certain students or for use in a mathematics club.

One possibility is to pursue the topic of finite geometry. After all, Postulates 1 through 9 do not rule out finite geometry. As a reference we suggest, Miniature Geometries by B. W. Jones in The Mathematics Teacher, February, 1959.

Another possibility is to study the following four problems. They lead to results that are much more striking than the theorems in the text. They could not be included in the text at this point since they assume, as later postulates will guarantee, that our geometry contains enough points and lines to make possible the constructions we describe. However, as exploratory problems they are very interesting. The relations suggested by the figures are quite unexpected and the "discoveries" suggested are actually theorems which can be proved, though we shall not do so. In exploring these problems you may find it appropriate to refer to pages 170-172 and 185-191 in What Is Mathematics? by R. Courant and H. Robbins, Oxford University Press, 1958.

1. Draw two intersecting lines, such as ℓ_1 and ℓ_2 in the following figure. On ℓ_1 choose three distinct points, A_1, B_1, C_1, and on ℓ_2 choose three distinct points, A_2, B_2, C_2.



37

These can be any points, as long as none of them coincides with O, the intersection of ℓ_1 and ℓ_2. Now draw lines $\overleftrightarrow{A_1B_2}$ and $\overleftrightarrow{A_2B_1}$ and call their point of intersection R. Also draw lines $\overleftrightarrow{A_1C_2}$ and $\overleftrightarrow{A_2C_1}$ and call their point of intersection Q. Finally draw lines $\overleftrightarrow{B_1C_2}$ and $\overleftrightarrow{B_2C_1}$ and call their point of intersection P. Can you see any interesting relation involving P, Q, and R?

Repeat the experiment

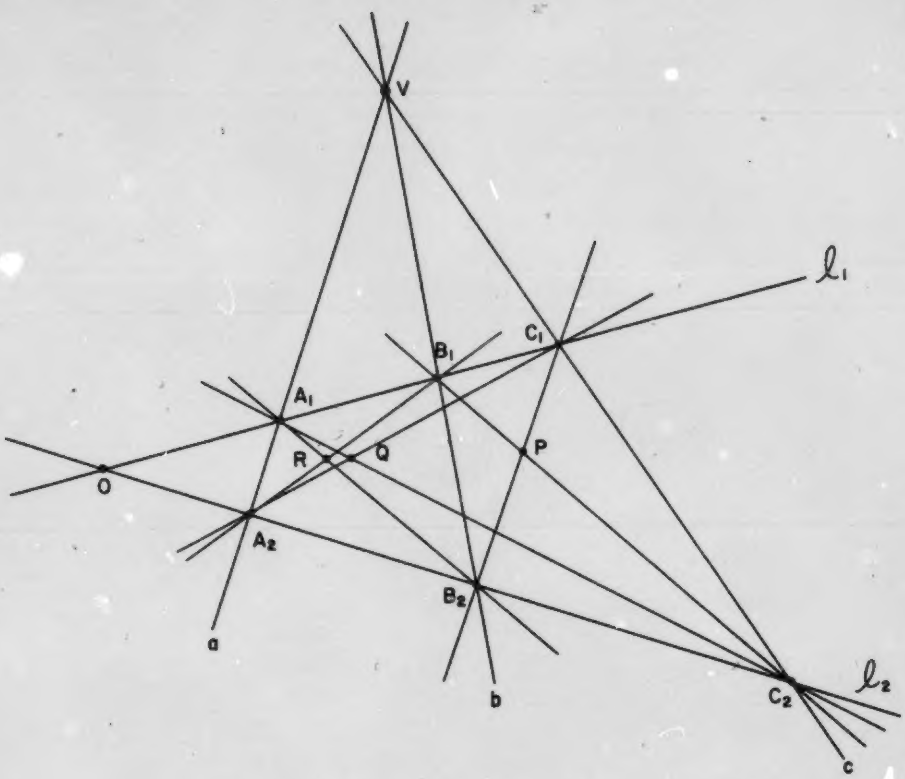
(a) using the last figure with A_1 renamed B_1, B_1 renamed C_1, and C_1 renamed A_1;

(b) with a new figure in which A_1, B_1, C_1 are on the same side of O but A_2 is on the opposite side of O from B_2 and C_2;

(c) with a new figure in which B_1 is on the opposite side of O from A_1 and C_1, and A_2 is on the opposite side of O from B_2 and C_2.

Do P, Q, and R always seem to be collinear? Do your examples prove this?

2. Now make your experiment a little more special. Instead of taking A_1, B_1, C_1 and A_2, B_2, C_2 anywhere on ℓ_1 and ℓ_2, respectively, select them as follows. In the plane of ℓ_1 and ℓ_2, let V be any point which is not on ℓ_1 or ℓ_2 and let a, b, c be three distinct lines passing through V but not through O. Let A_1 and A_2 be the points in which a intersects ℓ_1 and ℓ_2, respectively. Let B_1 and B_2 be the points in which b intersects ℓ_1 and ℓ_2, and let C_1 and C_2 be the points in which c intersects ℓ_1 and ℓ_2. Now find P, Q, and R just as you did in Problem 1. Do they still seem to be collinear?

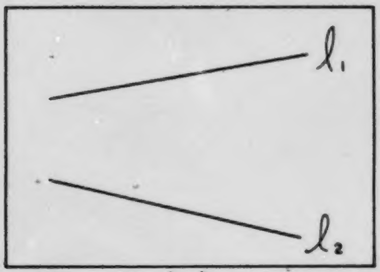


Would you expect them to be if your induction in Problem 1 were correct? Do P, Q, and R have any other interesting property which they did not have in Problem 1? Repeat this experiment three or four times, including at least one case in which V lies within the smaller of the angles formed by ℓ_1 and ℓ_2. Do P. Q, R and O always seem to be collinear? Do your examples prove this?

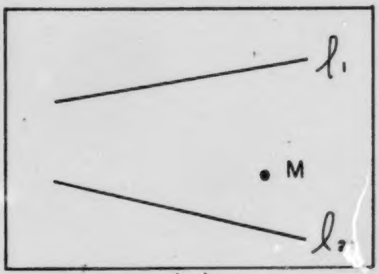
3. Extend the experiment described in Problem 2 as follows. Consider four lines, a, b, c, d, through V and let A_1, B_1, C_1, D_1 and A_2, B_2, C_2, D_2 be the intersection of the lines with ℓ_1 and ℓ_2, respectively. Using A_1, B_1, C_1 and A_2, B_2, C_2 find P, Q, R, just as before. Then using B_1, C_1, D_1, and B_2, C_2, D_2 in the same way, determine the points analogous to P, Q, R, say

S, T, U. What relation seems to be satisfied by the points P, Q, R, S, T, U, and O? If your induction in Problem 2 were correct could you have concluded by deduction, without performing the experiment, that P, Q, R, S, T, U and O would all be collinear? Why?

4. Given that under the conditions of Problem 2, the points P, Q, R, O are always collinear, show how this fact can be used to solve the following problem: Let ℓ_1 and ℓ_2 be portions of two lines drawn on a sheet of paper which is too small to contain the point of intersection of the lines, as in Figure (a).



(a)



(b)

By working only on the sheet of paper, draw a line which will surely pass through the intersection of ℓ_1 and ℓ_2. How could you draw the line determined by a given point, M, (Figure (b)) and the inaccessible intersection of ℓ_1 and ℓ_2?

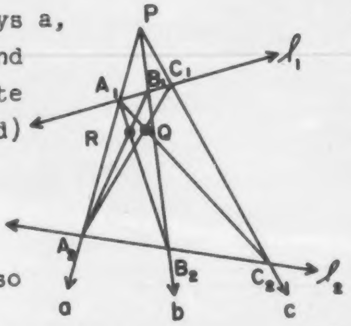
Concerning the solutions of these problems we offer the following.

1. P, Q, R always appear collinear. However, this is a generalization (an induction) from examples, and does not prove that P, Q, R will always be collinear.

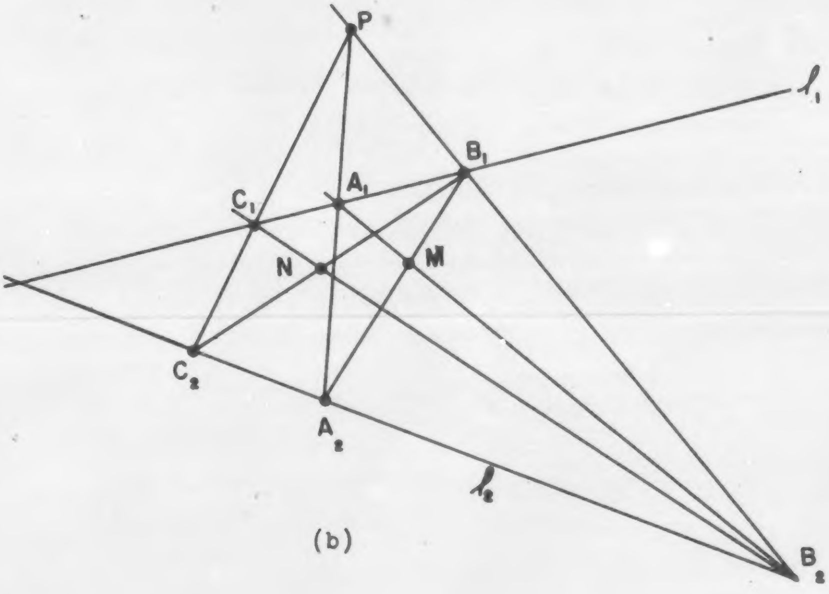
2. P, Q, R still appear collinear. We would expect them to. P, Q, R, O appear collinear in all the drawings, but this does not prove them collinear.

3. P, Q, R, S, T, V and O appear collinear. Yes, this could have been concluded by deduction from Problem 2 on the assumption that any three points determined in the manner of Problem 2 are collinear with O. In Problem 3, we have merely determined more than three such points.

4. From P, any point not on ℓ_1 or ℓ_2, draw three rays a, b, c, intersecting ℓ_1 and ℓ_2 as in Problem 2. Locate two points, (three if desired) corresponding to any two of the points P, Q, R of Problem 2. The line containing these points will also contain the point of intersection of ℓ_1 and ℓ_2. See our completion of Figure (a) at the right.



(a)



(b)

From any two points A_1, B_1 on ℓ_1, draw lines through M and call their intersections with ℓ_2, respectively, B_2 and A_2.

Let P be the intersection of $\overleftrightarrow{A_1A_2}$ and $\overleftrightarrow{B_1B_2}$.

Through P draw any line that intersects ℓ_1 and ℓ_2. Call these points of intersection C_1 and C_2, respectively.

Let N be the intersection of $\overleftrightarrow{B_1C_2}$ and $\overleftrightarrow{B_2C_1}$.

Then M, N are collinear with the inaccessible intersection of ℓ_1 and ℓ_2. See our completed Figure (b) above.

Illustrative Test Items for Chapter 2

1. Read each of the following statements carefully. If the statement is true as it stands, write "true". If it is false, write the word or phrase which, if substituted for the underlined portion, would change the statement to a true one.

Examples: 1. The United States is a set consisting of forty-eight states. fifty

2. The intersection of the sets {3, 2, 8} and {5, 9, 2} is the set {2}. true.

(a) The set {1, 3} is identical with the set {3, 1, 0}. ________________.

(b) The set denoted by {1, 2, 3, 4, ...} is the set of the first four natural numbers. ________________________.

(c) The solution set of the equation $(x - 3)(x - 1) = 0$ is {3, 1}. ____________.

(d) The intersection of two sets cannot be the null set, in accordance with the terminology usage agreed upon in our text. ______________.

(e) A set of five elements cannot be in one-to-one correspondence with a set of ten elements.

______________.

(f) The postulates we have studied thus far assure us that at least five points exist. _________.

(g) The points of a set are collinear if there is a plane which contains all of them.

______________.

(h) If a line intersects a plane which contains it the intersection is a single point. _________.

(i) When a biologist draws a conclusion from a series of observations of the habits of a fruit-fly he is reasoning deductively.

________________.

2. Consider the following sets:

A_1 is the set of all boys in the 10th grade.

A_2 is the set of all girls in the 10th grade.

A_3 is the set of all 10th grade geometry students.

A_4 is the set of all students in high school.

A_5 is the set of all 10th grade students.

A_6 is the null set.

Identify each of the following sets as one of the above sets.

(a) intersection of A_1 and A_5.

(b) union of A_3 and A_4.

(c) intersection of A_3 and A_4.

(d) union of A_1 and A_2.

(e) intersection of A_1 and A_2.

3. Write the number preceding the correct answer to each of the following in the space provided.

( ) (a) Generalizing from specific cases is called (1) deductive reasoning, (2) inductive reasoning (3) direct reasoning (4) indirect reasoning.

( ) (b) The term which is not an undefined term in our study of geometry is (1) point, (2) line (3) plane (4) space.

( ) (c) Statements logically deduced from basic initial assumptions are called (1) postulates (2) axioms (3) theorems (4) definitions.

( ) (d) The "if" clause of a statement written in the "if ..., then ..." form contains the (1) conclusion (2) proof (3) hypothesis (4) postulate.

4. Indicate whether the following statements are True or False.

(a) _____ A line and a plane always have at most one point in common.

(b) _____ Every two points are collinear.

(c) _____ A line has two endpoints.

(d) _____ If three points are collinear then they are coplanar.

(e) _____ A point and a line not containing it always lie in one and only one plane.

(f) _____ Given two points there are at least two planes which contain them.

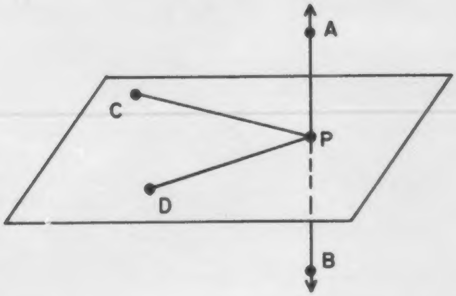
(g) _____ The set of rational numbers is a subset of the set of irrational numbers.

(h) _____ A counter-example verifies the truth of a general statement.

(i) _____ Undefined terms are not necessary for mathematical reasoning.

(j) _____ The empty set is a subset of every set.

5. Fill in the blanks in the statements below on the basis of the figure shown. IMPORTANT: If none of the points given satisfies the condition, write NONE in the blank space.



Points A, P, and _____ are collinear.

Points D, P, and _____ are collinear.

Points P, D, B, and _____ are coplanar.

Points C, A, B, and _____ are coplanar.

6. Indicate which part of each of the following statements is the hypothesis:

    (a) If three points lie in one plane, they are said to be coplanar.

    (b) If a set is empty, it contains no elements.

    (c) $3x - 7 = x + 2$ when $x = 5$.

    (d) The product of two integers is also an integer.

7. Rewrite each of the following statements in the "If...,then..." form.

    (a) A number greater than zero is positive.

    (b) Two lines which intersect in a single point are not parallel.

    (c) We stop when the light is red.

    (d) A line which does not lie in a plane can intersect that plane in only one point.

    (e) $(a - b)^2$ is positive whenever a and b are real numbers such that $a \neq b$.

    (f) A four-sided polygon is called a quadrilateral.

    (g) Squares are rectangles.

8. Underscore the hypothesis in each of the statements you wrote in "If ..., then ..." form in problem 7.

9. Fill in the blanks for the proof of the theorem: "Space contains at least two planes."

   (a) From a theorem, we know that there are three noncollinear points.

   (b) There is just _____ plane containing these points.

   (c) There is a _____ not in this _____.

   (d) This _____ is contained in at least one _____.

   (e) These two _____ are distinct since there is a _____ in one not in the other.

   (f) Therefore, space contains at least _____ planes.

10. Fill in the blanks with one or more words which will correctly complete the proof of the theorem: "If two distinct lines intersect, they intersect in exactly one point."

   (a) Suppose that p and q are two distinct intersecting lines. Since they intersect, they have at least __________ in common which we shall call A.

   (b) Either they have another __________ in common or they do not.

   (c) Suppose there is another __________ which we shall call B that lies in p and q.

   (d) Then there is exactly __________ containing A and B.

   (e) This means that p and q are __________.

   (f) By hypothesis we know that p and q are __________ lines.

   (g) Since the statement in (e) __________ the statement in (f), we rule out the possibility that p and q have __________ in common.

   (h) Therefore, we conclude that p and q have __________ in common.

11. Prove that, in any city whose population exceeds 500,000, there are at least two people who have the same number of hairs on their heads. (Assume that the maximum number of hairs per head is 300,000.)

12. You are "supermind". You have the following "postulates" and "definitions". Write three theorems that can be deduced from these.

Definition. BIMTON is the set of all BIMS.

Postulate 1. There are at least two BIMS in BIMTON.

Postulate 2. Every HUT is a set of BIMS and contains at least two BIMS.

Postulate 3. Given two distinct BIMS, there is one and only one HUT that contains them.

Postulate 4. No single HUT contains all the BIMS in BIMTON.

Chapter 3

DISTANCE AND COORDINATE SYSTEMS

In this chapter the text departs from the traditional development of Euclidean geometry by postulating relationships between the real numbers and points in space and between the real numbers and points on a line. There are good reasons for postulating such connections. Some of these we have explained in the text, while the students' lack of mathematical background makes it inappropriate to mention others. The easiest way to bring out these latter reasons is to quote from several authors. We first quote two passages (pages 191 and 207) from Elementary Mathematics from an Advanced Standpoint, Vol. II (Geometry), which was written over 50 years ago by Felix Klein, an outstanding mathematician and devoted student of geometry:

> The Greeks possessed no independent arithmetic or analysis, neither decimal fractions, which lighten numerical calculation, nor general use of letters as symbols in reckoning. Both of these are inventions of early modern times, during the Renaissance. As a substitute, the Greeks had only a calculus in geometric form, in which operations were performed constructively with segments or other geometric magnitudes, instead of with numbers, a process much more cumbersome than is our arithmetic. Coupled with this also is the fact that the Greeks did not have negative and imaginary numbers, which are really what give facility to our arithmetic and analysis. Consequently they lack the generality of method which permits the inclusion in a formula of all possible cases. A tedious distinguishing of cases played a great role with them. This lack is often very noticeable in geometry, whereas today, by employing analytic aids, we can easily achieve complete generality, and we can avoid all distinction of cases.

. . . .

> As a matter of fact, the source of all fundamental geometric concepts and axioms is our naive geometric perception. From it we choose the data which, in appropriate idealization, we lay at the base of the logical treatment. As to which choice should be made, however, there can be no absolute judgment. The freedom which exists here is subject to only one restriction, namely, the requirement that the system shall fulfill its purpose of guaranteeing a consistent development of geometry.
>
> Another observation concerns our attitude to anayltic geometry, and our criticism of certain traditions, from Euclid on, which have long since ceased to conform to the position of mathematical science, and which should, on that account, be given up in school instruction. In Euclid, geometry, by reason of its axioms, is the rigorous foundation of general arithmetic, including also the arithmetic of irrational numbers. Arithmetic remained in this position of bondage to geometry well on into the nineteenth century, but since then there has been a change. Today arithmetic analysis, as a proper fundamental discipline, has reached a dominating place. This is a fact which ought to be reckoned with in the development of scientific geometry.

Oswald Veblen, a member of the Institute for Advanced Study for many years, was an ardent student of geometry. He wrote in an article, The Modern Approach to Elementary Geometry [The Rice Institute Pamphlet, Vol. XXI, No. 4, October, 1934, pp. 209-221] the following:

> I do not advocate the use of any particular set of axioms in the schools but I do advocate the introduction of analytic methods in elementary geometry.

. . . .

> Obviously, the development of the propositions of geometry from these foundations [here he is referring to axioms similar to ours] should be closely related to the study of elementary algebra, linear and quadratic equations, and the like. Moreover, I ought to guard against one possible misunderstanding. The working out of this program does not mean the elimination of synthetic proofs from geometry. There are plenty of cases in which a synthetic or a mixed proof is easier than a purely analytic one. In such cases I would use the simplest and most desired process which I knew. The result would be, I am confident, that the student would have as good a grasp of synthetic methods as at present, and a much better idea of what it is all about.

Our text has endeavored to develop the thesis presented by the authors quoted. In this chapter we postulate connections between real numbers and points in space and between real numbers and points on a line. Because of the active role which numbers play in our development of geometry, we devote Sections 3-2 and 3-3 to a review of the real numbers, emphasizing features that will be particularly important in the sequel. In Section 3-4 we introduce the notion of distance. In Section 3-5 we introduce coordinates on a line. The remainder of the chapter (Sections 3-6 through 3-11) develops further the connections between real numbers and geometry. This is followed by a brief summary of the chapter in Section 3-12. To summarize, the chapter is divided into two clear-cut parts for teaching purposes:

I. Sections 3-1, 3-2, and 3-3. Introduction and review of real numbers emphasizing features important to our development.

II. Sections 3-4 through 3-12. Statement of postulates connecting real numbers and geometry. Development of consequences of these postulates followed by a summary of the chapter.

Let us discuss the teaching problems and the content of these divisions in more detail.

In Division I, the most important features of the real number system for our later development are reviewed. These are summarized by the word order. We develop the topic of inequalities. Except for the names of the order properties which are used later in the text, the short unit devoted to inequalities may be largely review for some classes and will require little class time; some classes may omit it entirely.

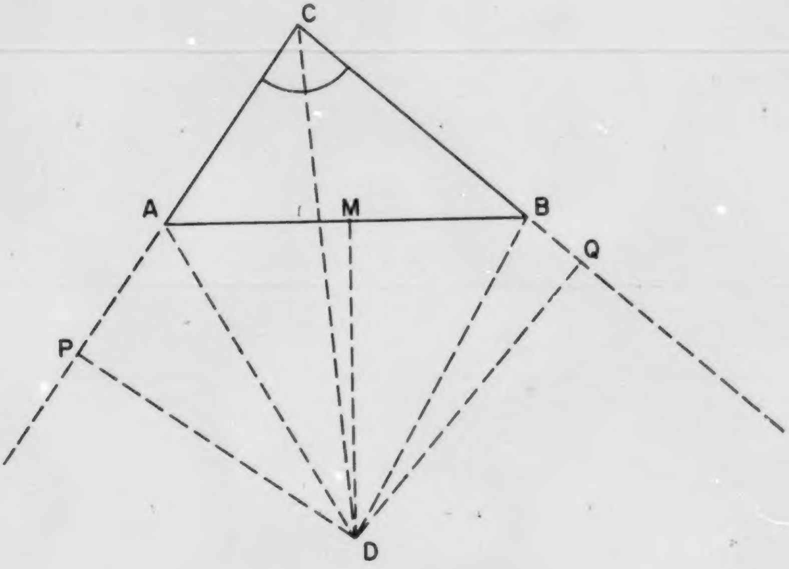
In Division II, the central ideas developed in our postulates are the notion of distance and the concept of a coordinate system. In terms of these ideas we define ray, segment, between, and midpoint, and explore the relationship between different coordinate systems. Our exposition as measured in numbers of pages may seem lengthy, but it is well to realize that in writing a text we cannot make the sort of informal and informative remarks that can be made in the classroom where the teacher can tell by the student response whether he has been understood or not. If a teacher senses that he is not understood, he can easily add sufficient remarks to clarify his original statement. Neither does an author have the ability to point, using a gesture, as a teacher can at a chalkboard. "Pointing" for an author may involve several paragraphs or pages. We can afford, however, to be much more informal in this commentary and intend to suggest appropriate informal classroom remarks you can use to explain the material in the text. We also depend on you to replace the paragraphs of "pointing" by appropriate gestures at the chalkboard.

In presenting Postulate 10 concerning distance, as well as later postulates, it is important to note that we are continuing our description of spatial relationships and that we are not trying to tell how to measure distance. Roughly speaking, all we are saying at this stage is that in the measurement of distance, however accomplished, the end result is a number and that in the background there is some fundamental unit of measurement. It would be most confusing to students to give the impression that the postulate says that we can assign any positive number we like to each pair of distinct points as long as we assign 1 to the pair of points A and A'. This is not the case. This postulate is only a part of our description of the concept of distance and does not contain instructions for doing anything. Indeed, this description will not be completed until we have deduced the Pythagorean Theorem in Chapter 7. The important thing is to get students to see that any reasonable description of distance must contain at some stage a statement like that in Postulate 10.

Postulate 11 is placed in Section 3-4 not only because it is an obvious improvement to the description of the concept of distance but in addition because it is needed in Section 3-6. It also enables us to do some preliminary work with relations between coordinate systems to prepare the student for the full treatment in Sections 3-9 and 3-10 which is based on Postulate 13. Indeed it can be seen that Postulate 13 implies Postulate 11. In a mathematical treatise it might be considered undesirable to have a later postulate imply an earlier one. However, in a classroom text, we feel there are strong pedagogic reasons for doing this.

Another aspect of this chapter is the introduction of the concept of betweenness. This idea, though intuitively natural, is one that has rarely been formalized in high school treatments of geometry. In contrast to the abstract synthetic treatment of betweenness given in many treatises, our introduction of real numbers into geometry makes it easy to be clear and precise about this notion. Indeed, we have two main reasons for using the real numbers: These are, (1) to introduce the notion of betweenness and (2) to make possible the blending of synthetic and coordinate geometry. While we are quite explicit about the use of betweenness in this chapter, you will find places later in the text that technically need explicit use of the notion to establish the relative position of points, but that this is not pointed out to the student. We ignored these technicalities because we feel that the student not only will not notice the omission but might become confused if we tried to point it out. On the other hand it might be appropriate, toward the end of the school year to discuss with a good class the fallacy in the proof that every triangle is isosceles. The "proof" goes as follows:

In a triangle ABC let $\overrightarrow{CD}$ be the bisector of the angle at C. Let M be the midpoint of $\overline{AB}$, and let $\overline{MD}$ be in the perpendicular bisector of segment $\overline{AB}$. Suppose further that D is the point of intersection of the angle bisector and the perpendicular bisector.



Also let P and Q be at the foot of the perpendiculars from D to $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$, respectively. Now AD = BD because D is on the perpendicular bisector of $\overline{AB}$. Also DP = DQ because D is on the angle bisector. Moreover $\angle P$ and $\angle Q$ are right angles. Hence $\triangle APD \cong \triangle BQD$. Consequently, AP = BQ. We see that the right triangles CPD and CQD are congruent since DP = DQ and their hypotenuses are in common. Thus CP = CQ. But CP = CA + AP and CQ = CB + BQ. So, since AP = BQ, and CP = CQ, it follows that CA = CB. Hence the triangle ABC is isosceles.

If this argument is examined closely it will be seen to be incomplete since we used relative positions of the points as shown in the diagram without proving that the positions indicated were correct.

Unless we have a definition of betweenness, it is impossible to speak of the relative position of points, and we have no way of justifying statements like $CP = CA + AP$. If we do have a theory involving betweenness, we can prove that, for a non-isosceles triangle, the points do not have the relative positions indicated in the figure. (Incidentally, it is almost impossible to give a "false" proof of this type analytically since the equations of lines and the coordinates of points carry with them inherently all the "betweenness information" that exists relative to them.)

In addition to introducing betweenness we also begin in this chapter to use the real numbers to blend synthetic and analytic geometry. At this stage, we are concerned mainly with coordinate geometry of the line. The two key theorems are Theorems 3-5 and 3-6 which are proved in Sections 3-9 and 3-10 following the statement of Postulate 13 in Section 3-8. Theorem 3-6 has a key role in our development of coordinate geometry in the plane and in space in Chapters 8 and 9.

It may seem surprising that there is no discussion in the text of straightedge and compass constructions for midpoints of segments, for dividing a segment into an integral number of congruent parts, etc. Constructions such as these have no formal part in our development of geometry. The geometric theory on which they are based does have a part. For instance, in our treatment of circles in Chapter 12, it is appropriate to give problems like the following: Given two coplanar circles of equal radius that intersect in two points, show that the line determined by these points of intersection is a perpendicular bisector of the segment joining their centers. Such a problem can be followed by a classroom comment on the associated construction. The fact that the Greeks had only a calculus in geometric form, in which operations were performed by geometric constructions, has already been commented on in the quotation from F. Klein at the beginning of these remarks. For us, numbers are not segments. Our conceptions of the roles of arithmetic and geometry is not that the first depends on the latter or vice versa but that the two have equal status and it is desirable to relate them. Coordinate geometry is precisely this strong relationship between them. Although constructions have no formal part in our development, there is no reason to exclude them from physical geometry. We encourage the teacher to use them as appropriate at the chalkboard. Appendix 12 on the use of straightedge and compasses has been included primarily for reference purposes. If desired, it can be used as a review type unit (with a fresh flavor) near the end of the course.

We conclude our general remarks about this chapter by observing that the emphasis is largely on lines and that we need in later chapters to extend our ideas to planes and space. In particular, our description of distance is incomplete. For instance, given three noncollinear points A, B, C, and AB and AC (relative to some unit-pair), we cannot, at this stage, show that $AB + AC > BC$. We also remark before beginning our paragraph by paragraph comments that as in Chapter 2 the emphasis should be on understanding, not on requiring students to repeat or create formal proofs. Furthermore, the development in Chapter 4 parallels that of this chapter. This parallelism will help students to clear up minor points in Chapter 3 while doing Chapter 4. Thus, we recommend that as soon as the students have learned the material in this chapter moderately well that they go on with Chapter 4.

56 The idea of a number line is closely related to the notion of a coordinate system on a line as introduced in Section 3-5. This is also commented on in the Text, page 57. It is introduced at this stage for these reasons:

(1) to make easier the exposition of the notions of order;

(2) to emphasize the intuitive background of a coordinate system on a line.

57 Some students argue that π is approximately 3.14, and since $\frac{22}{7}$ is 3.14 to the nearest hundredth, π must be $\frac{22}{7}$. This is like the fallacious argument: yellow is approximately chartreuse, and green is approximately chartreuse, so yellow is green. The teacher needs to remind the student that the approximation is rational, but π is not.

57 The reference to Appendix II and the accompanying statements about the postulational nature of algebra emphasizes something that might be new to some students. The unfortunate practice of waiting until we get to geometry before considering deductive systems leaves the student feeling that geometry is where proof starts and ends. He is often unaware that algebra can be described using a relatively small collection of postulates. While time will not permit consideration of the postulational development of algebra, we are going to assume that the basic algebraic properties are understood by the student.

57-64 We use $>$ exclusively in our discussion here, in Problem Set 3-2, in the first part of Section 3-3, and in Problem Set 3-3a. We did this to avoid confronting the student with several symbols at once since this might interfere with out presentations of properties on page 63 and their relationships to the number line. That is, the symbols are not our major concern at this stage but the concept of order among real numbers is.

64 Here, we begin introducing the symbol $<$. This is followed by the symbols $\geq$ and $\leq$.

69 More precise standard units (in the sense that they do not vary with temperature, etc.) have now been suggested by nuclear physicists in terms of the wave length of light emitted from some very pure isotopes.

70 As we stated in our general remarks we remind you that Postulate 10 should be taken as a first formal statement in our description of the concept of distance, not as a rule of procedure for assigning measure of distance to every pair of points in space. It should be admitted quite

frankly that it is an incomplete statement of ideas about distance. It essentially is the statement that the end result of measurement however performed is a number and that in the background somewhere there must have been some unit in terms of which the measurement was made.

71 In connection with the notation for the distance between P and Q relative to $\{A,A'\}$ note that

$$\begin{aligned} PQ \text{ (relative to } \{A,A'\}) &= QP \text{ (relative to } \{A,A'\}) \\ &= PQ \text{ (relative to } \{A',A\}) \\ &= QP \text{ (relative to } \{A',A\}) . \end{aligned}$$

This is because no order between the points A,A' which form the set $\{A,A'\}$ and no order between the points P, Q is implied by Postulate 10.

72 Postulate 11, just as Postulate 10, should be considered as a small part of our description of distance. The problems in the problem set have been designed deliberately not only to reinforce the ideas presented in this section but to strengthen the intuitive background for coordinate systems on a line. For this reason there is considerable reference to and dependence on the idea of a physical ruler in these problems.

76-77 The intuitive background on which the definition of a coordinate system is based may be so strong with some students that they may not realize that there is a real need to question whether such a correspondence exists and to postulate the existence as is done in Postulate 12. The question suggested is the question of the need to assert this correspondence in our formal geometry. We intend to cast no doubt whatsoever on our belief that this is an important part of our ideas about points and lines and their relationships to the real numbers. The point is that a definition doesn't play the role of a postulate. A mathematical definition does not assert the existence of the entity defined. You may define the pot of gold at the end of the rainbow with great precision but you may experience great disappointment if you start to search for it and it does not exist.

In classes that are familiar with absolute value, the last sentence in the definition of a coordinate system could be replaced by the sentence:

> If numbers r and s correspond to points R and S, then $|r - s|$ is the same as the number RS (relative to $\{A,A'\}$).

For such classes this modification of the text is desirable and the absolute value concept should be used throughout the rest of the course. We avoided the introduction of the absolute value concept until Chapter 8, so as not to interfere with the understanding of this part of geometry. See also Problems 5(e) and 10, pages 80 and 81 of the text.

In using the Ruler Postulate, it is important to note the necessity of doing several things:

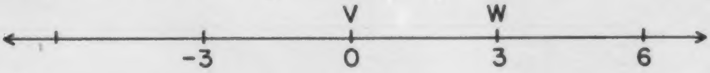
(1) Identify the unit-pair.

(2) Specify the line to which the Postulate is being applied.

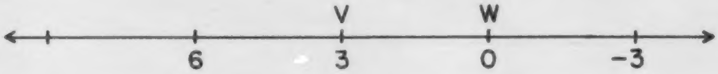
(3) Identify the point that is to play the role of P in the Postulate, and the point that is to play the role of Q.

Later, when we use the same unit-pair throughout a problem, item (1) is often not explicitly stated since it is well understood what unit-pair is being used. Item (2) can be suppressed since the points that play the role of P and Q must be mentioned and they determine the line in question. Item (3) cannot be suppressed in general. It contains what might be the placement feature of the postulate (we will encounter a similar placement feature in connection with the Protractor Postulate in the next chapter). Let us explain this feature in the same way that we suggest explaining it to students, namely, by use of a diagram. Suppose that we have a unit-pair $\{A,A'\}$ and a pair of points V,W such that VW(relative to $\{A,A'\}=3$. If we let V play the role of P, W play the role of Q

in the Postulate, then coordinates on $\overleftrightarrow{VW}$ are established as shown in part in the diagram below:



If, however, we let W play the role of P and V play the role of Q, then our diagram is as shown below:



The student should notice that in the first case, it is as if a ruler were placed against the line with 0 at V and 3 at W. In the second case, the ruler is "reversed" and 0 is at W and 3 at V. Instead of saying in the first case that "V plays the role of P" it would be equally suitable to say that V is taken as origin. Similarly, in the second case, W is the origin.

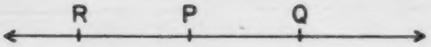
80-81 In connection with the remark following Problem 5 about Problem 10, we have avoided absolute value at this stage to prevent making difficulties for students not already familiar with it. We should encourage those familiar with it to use it.

83 The fact that a point on a line separates it into two parts has already been suggested in Problem 4 in the preceding Problem Set.

84 Note that our definition of ray $\overrightarrow{AB}$ is in terms of a particular coordinate system on $\overleftrightarrow{AB}$ in which A is the origin and B is the unit-point. In Section 3-10 we will show how to recognize a ray in terms of any coordinate system on the line containing it. At this stage we cannot prove that if C is in $\overrightarrow{AB}$ and if C is not A then $\overrightarrow{AC}$ is the same ray as $\overrightarrow{AB}$. This proof depends on the ideas developed in Section 3-10. We do not point this out to the student at this early stage of our development since we think this remark would be more confusing than helpful. Further, we have tried to arrange our presentation so this issue will not arise. On the other hand, by our example

of choosing a special coordinate system for this definition as well as that for segment, we hope to emphasize our freedom to choose coordinate systems conveniently. This freedom will be exploited in connection with proofs using coordinates in the geometry of the plane and space.

84 In the proof of Theorem 3-2, the relationship between the old coordinate x and new coordinate x' is given as $x' = -x$. The derivation of this result follows the idea suggested in Problem 6 in the previous Problem Set. Let us consider the dependence of this result on Postulate 11 in more detail. Suppose the situation is as depicted below:



-1	0	1	old coordinates
1	0		new coordinates

Since R has coordinate -1, $PR = 1$. Hence, by Postulate 11, distances measured in the old coordinate system are exactly the same as those measured in the new coordinate system. Consider now a point T on $\overleftrightarrow{PQ}$ whose old coordinate is 5. What is the new coordinate x'? We know, measuring in the old coordinate system, that $RT = 6$ and $PT = 5$. Measuring in the new coordinate system, we know that $RT = 1 - x'$ or $RT = x' - 1$ and that $PT = 0 - x'$ or $PT = x' - 0$. This choice exists since we do not know the sign or relative size of x' without relying on the diagram. If $PT = x'$ then since $PT = 5$, we have $x' = 5$. But then, $RT = 5 - 1 = 4$ which is impossible. Hence, $PT = -x'$ and consequently $x' = -5$.

Consider a more general situation. Suppose that the point T on $\overleftrightarrow{PQ}$ is such that its old coordinate x is positive. Suppose its new coordinate is x'. We know that $RT = x - (-1) = x + 1$ and that $RP = x$ where we measure in the old coordinate system. Measuring in the new coordinate system, we know that $RT = x' - 1$ or $RT = 1 - x'$ and that $RP = x'$ or $RP = -x'$. The choice exists, since we cannot decide without reading into our diagram, whether x' is positive or negative. However, we can resolve this difficulty without relying on the diagram. Suppose $RT = x' - 1$. Then, from $RT = x + 1$, we have $x + 1 = x' - 1$ or $x' = x + 2$. But, then substituting this value of x', we have $RP = x + 2$ or $RP = -(x + 2)$ both of which disagree with the fact that $RP = x$. Hence, $RT \neq x' - 1$. Therefore, $RP = 1 - x'$. But, then $1 - x' = x + 1$ or $x' = -x$, which we wished to show. Similarly, we can argue that $x' = -x$, if x is negative and certainly $x' = -x$ if $x = 0$. There is no need to go through the above with students. They will probably be satisfied if you observe from a diagram that it is clear that the sum of the old and new coordinates is zero. A treatment of the general argument of the formal type above is given in Section 3-9, for interested students. We do not recommend it for all students.

86 It is technically necessary to raise the question in connection with the definition of segment as to whether one gets a set for the segment $\overline{AB}$ on the line $\overleftrightarrow{AB}$ if one uses A as origin and B as unit-point that is different from the set obtained if A is taken as unit-point and B as origin. We do not suggest raising this point with students. Our talk and problems in the text are deliberately arranged to suggest that they are the same. The fact that they are the same is settled in Section 3-10 although, again, no reference to the possible question is raised. The matter can be settled here, since, as brought out in the example, pages 86 and 87, the relationship between the two coordinate systems is given by $x' = 1 - x$. Consequently, if $0 \leq x' \leq 1$, then $0 \leq 1 - x \leq 1$. Thus, subtracting one, $-1 \leq -x \leq 0$ and on multiplying by minus one, $1 \geq x \geq 0$. Similarly, starting with $0 \leq x \leq 1$, we can deduce $0 \leq x' \leq 1$. Thus, which point is taken as the origin and which is taken as the unit-point in applying this definition is immaterial, since either way yields the same set for the segment.

91 The notion of betweenness may not have been used in formal geometry that you have taught before. It was found toward the end of last century by Hilbert and others to be an essential part of Euclidean geometry that had been omitted from the formal development although used informally and in an essential way. In short, it was read from diagrams and figures unconciously. Reference to Hilbert and his work can be found in most histories of mathematics. A list of some histories occurs at the end of Chapter I of the text.

91 If we say that P, Q, R, S are collinear in that order, it also follows that each of Q and R are between P and S.

93 The problems in Problem Set 3-7 in addition to providing drill on ideas in this section are designed to motivate Postulate 13 in the next section.

96 Postulate 13 further improves the formal description of our ideas about distance by expressing the relationship between distances measured relative to different unit-pairs. It could have been stated that given two unit-pairs $\{A,A'\}$ and $\{B,B'\}$ there exists a non-zero constant k such that for any two points P,Q

$$PQ \text{ (relative to } \{A,A'\}) = k \cdot PQ \text{ (relative to } \{B,B'\}).$$

In this form P and Q do not have to be distinct. Also, it would be fairly natural to call k the scale factor relating the two choices of the unit-pair. Indeed, if P, Q are chosen to be B,B', we see that

$$BB' \text{ (relative to } \{A,A'\}) = k$$

since

$$BB' \text{ (relative to } \{B,B'\}) = 1 .$$

In the case that

$$BB' \text{ (relative to } \{A,A'\}) = 1,$$

we see that

$$PQ \text{ (relative to } \{A,A'\}) = PQ \text{ (relative to } \{B,B')$$

which is essentially the statement of Postulate 11.

It should be pointed out to students that many of the problems in the previous problem set involved computing the quotient expressed in this Postulate and using the fact that it is constant.

99 The starred problems in this set are designed to motivate the analysis of the relationship between different coordinate systems on a line in the next section.

102-103 From the fact that the distance in C' between any two points is 3 times the distance in C between the same two points and using the obvious sense of direction indicated in the diagram, it is readily apparent that the coordinate of t' is -30. The point to the computation is that without using the obvious sense of direction indicated in the diagram, the same conclusion can be reached after eliminating various possibilities. In formal geometry, we should not have to depend on a diagram. Problems 1 and 3 of Problem Set 3-9 require the student to make similar computations.

While we believe that all students should be able to do these computations, we do not feel that the algebraic analysis of the problem for determining x' is suitable for most students. This algebraic analysis, using constants r and s for the coordinates of R and S in C instead of 4 and 6, and r' and s' for their coordinates in C' instead of -3 and -9, leads by exactly the same method of proof to the general theorem, called the Two Coordinate System Theorem. We do not feel that the student is cheated if the algebraic treatment is avoided since the method of analysis is exactly that of the numerical example. It consists in systematically excluding cases.

The Two Coordinate System Theorem has two important aspects. The first is that in a sense it says that you will not get into trouble in doing problems such as the numerical one in the text by shortening your work by reading "direction" from a suitable diagram. Most people do such computations this way, anyway. The other aspect is that of the simple, first degree (that is linear) relationship that exists between x in C and x' in C'. After this theorem has been established we have another way of doing the computation for t' as follows.

From the theorem, we know that the relationship between x' and x is given by

$$x' = ax + b$$

for some choice of constants a, b. Since the coordinate of R is 4 in C and -3 in C', we know that the constants a, b must be such that

$$-3 = a4 + b \ .$$

Similarly, using the coordinates of S, we know that

$$-9 = a6 + b \ .$$

Taking the "difference between these two equations" we find

$$6 = -2a \ ,$$

that is, $a = -3$. (Note that this is related to the ratio of distances between the C and C' systems). Since $a = -3$, it is easy to determine b from either of the two equations. We find that $b = 9$. (Note that this is related to the distance between the two origins in C.) We now see that x' and x are related by

$$x' = -3x + 9 \ .$$

Thus, in particular, if $x = 13$, then $x' = -30$ so that the c' coordinate t' of T is -30.

The use of the Two Coordinate System Theorem is brought out for the student in Problems 2, 4, and 5(b) of Problem Set 3-9.

It is worth noting that examples of linear relations between two coordinate systems have been obtained in the text on Pages 84 and 87 as well as in Problems 6, 7, 8, and 9 of Problem Set 3-5.

Problem 7 in Problem Set 3-9 is designed to bring out the significance of a and b in $x' = ax + b$ as well as why a cannot be zero. Problem 8 in the same set prepares the student for another form of the Two Coordinate System Theorem known as the Two-Point Theorem.

108 The Two-Point Theorem is important for the development of the theory in Chapter 8. Students should understand this theorem well before beginning Chapter 8.

Notice that the effect of the formula in Theorem 3-6 is that if the k coordinate of a point is zero, then its x coordinate is x_1, and if its k coordinate is *one*, then its x coordinate is x_2. Thus, by this theorem, if the origin and unit-point in one coordinate system (k) have the coordinates x_1 and x_2 in another coordinate system (x) then for any point, its x and k coordinates are related by

$$x = x_1 + k(x_2 - x_1)$$

109 Using our theorems so far established, we can recognize as indicated in the table, segments, rays, interiors, opposite rays, midpoints in any coordinate system. In particular, since we can recognize interiors of segments in any coordinate system, we can recognize betweenness in any coordinate system. This is so important it is stated as Theorem 3-7.

115 The idea of congruence is also introduced in connection with angles and, of course, for triangles. There is a discussion of the general notions in the Talk entitled *The Concept of Congruence*. It is also discussed briefly in the earlier Talk entitled *Equality, Congruence, and Equivalence*.

116 The content of the Point Plotting Theorem should be intuitively obvious to the student.

117-118 As remarked in the text, the addition property of distance on a line actually characterizes betweenness. In some texts the definition of betweenness is given in terms of this property. Note also the connection with this theorem and our earlier remarks about the "proof" that every triangle is isosceles.

121-123 Students should be encouraged to read this summary carefully and to note the vocabulary list.

Illustrative Test Items for Chapter 3

Part I

1. Write each of the following as an inequality.
   (a) p is a positive number.
   (b) k is a negative number greater than -10.
   (c) x is a number between 10 and 20.
   (d) r is not a positive number

2. Complete the blanks in each of the following and state the order property that applies.
   (a) If a, b, and c are real numbers such that $b > a$ and $a > c$, then ____ > ____. Why?
   (b) If x is a real number and $-x > 4$, then x ___ -4. Why?
   (c) If x, y, and a are real numbers and $x > y$, then $x - a$ ____ $y - a$. Why?

3. Find the solution set for each of the following inequalities:
   (a) $3x + 14 > 5$
   (b) $x + 7 \leq 3x - 2$

4. Graph each of the solution sets in Problem 3.

5. If A corresponds to 0 and B to 1 in a coordinate system on a line, what set of numbers corresponds to:
   (a) $\overrightarrow{AB}$ (b) $\overrightarrow{BA}$ (c) $\overline{AB}$
   (d) ray opposite to $\overrightarrow{AB}$
   (e) ray opposite to $\overrightarrow{BA}$

6. Let the unit-pair for measuring distances be fixed in this problem. On a line, let a coordinate system be given. Let R and S be the points with respective coordinates -5 and 6. Suppose that another coordinate system on $\overleftrightarrow{RS}$ is chosen so that the new coordinate of R is 0 and the new coordinate of S is positive. What number is the new coordinate of S?

7. Three towns, Lander, Manton, and Amity, are collinear but not necessarily in that order. It is 9 miles from Lander to Manton and 25 miles from Manton to Amity.
   (a) Is it possible to tell which town is between the other two?
   (b) Which town is not between the other two?
   (c) What may be the distance from Lander to Amity?
   (d) Illustrate with sketches.

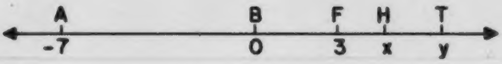
8. Given A , B , C are three collinear points with AB = 8 and CB = 5. If, also, the coordinate of B is -2, and the coordinate of A is less than that of C , what are the coordinates of A and C ? Draw two sketches giving different sets of answers.

9. Arrange the five distinct collinear points F , L , M , S , T in proper order if:

$$LM + MF = LF$$
$$SF + FT = TS$$
$$LS + SM = ML$$

10. Supply the appropriate missing symbols over each letter pair.
   (a) AB has no endpoints.
   (b) The endpoints of MR are M and R.
   (c) RQ has one endpoint, R.

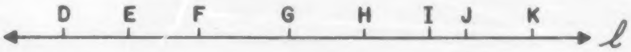
11. In the diagram A , B , F , H , T are collinear in that order and have coordinates as indicated.



   (a) The length of $\overline{AB}$ is _____.
   (b) The length of $\overline{AH}$ is _____.
   (c) The length of $\overline{BT}$ is _____.
   (d) The length of $\overline{FT}$ is _____.
   (e) The length of $\overline{HT}$ is _____.

12. If PQ (in zilks) is 3 times PQ (in flips), find:
   (a) RS (in flips), if RS (in zilks) = 1.
   (b) AB (in zilks), if AB (in flips) = 12.

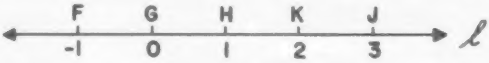
13. Given the following points collinear in the order shown in the diagram. Name the intersection of each listed pair of sets. (The answer to part (a) is $\overline{KG}$.)



(a) $\overrightarrow{GK}$ and $\overrightarrow{KG}$
(b) $\overrightarrow{GK}$ and $\overrightarrow{GE}$
(c) $\overrightarrow{GK}$ and $\overrightarrow{HG}$
(d) $\overrightarrow{ED}$ and $\overrightarrow{FG}$
(e) $\overrightarrow{FH}$ and $\overline{FH}$
(f) $\overline{GK}$ and $\overrightarrow{GK}$
(g) $\overline{GK}$ and $\overline{EH}$
(h) $\overrightarrow{GK}$ and $\overrightarrow{GI}$

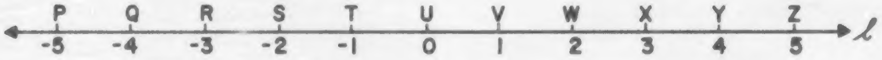
14. Name the union of the two sets indicated in each of parts (a), (e), (g), and (h) in Problem 13.

15. Given a coordinate system on line ℓ as pictured in the diagram:



Tell which of the following sentences are not meaningful in our development of geometry. For each meaningful sentence state whether it is true or false.
(a) $HG = 1$
(b) $\overline{FJ} = 4$
(c) $FK = 5$
(d) $\overline{HJ} \cong \overline{GK}$
(e) $\overline{HJ} = GK$
(f) $\overline{HJ} \cong GK$
(g) $HJ \cong GK$
(h) $HJ = GK$
(i) $JK = KJ$
(j) $\overline{JK} \cong \overline{KJ}$
(k) The length of $\overrightarrow{FJ} = 4$.
(l) The length of $\overline{GK} = 1$.

16. Given a coordinate system on ℓ which assigns numbers to points as indicated in the diagram.



Find each of the following measures:

(a) VU (relative to {U,V}) =

(b) WX (relative to {U,V}) =

(c) VZ (relative to {T,V}) =

(d) VX (relative to {V,Z}) =

(e) PR (relative to {P,Y}) =

17. Using the diagram in Problem 16 , give the new coordinate of Z , in a coordinate system whose origin and unit-point are:

(a) U and Z respectively

(b) U and W respectively

(c) T and W respectively

(d) W and T respectively

(e) S and P respectively

18. Which of the following inequalities represents the coordinates of a set of points known as a) a ray, b) a segment, c) the interior of a ray, d) the interior of a segment.

(1) $x > 2$

(2) $x \geq -1$

(3) $2 \leq x \leq 2.5$

(4) $x < 4$ or $x > 3$

(5) $2 < x < 5$

(6) $x < 4$ and $x > 5$

19. What is the coordinate of the midpoint of $\overline{AB}$ if the coordinates of the endpoints are -7 and 3 ?

20. Given M is the midpoint of $\overline{PQ}$. Find the coordinate of Q if the coordinate of P is 3 and the coordinate of M is -1.

21. Let A , B , P have coordinates -1 , 7 , p respectively. Find all the numbers that p might denote if:

(a) AP = AB

(b) AP = 3AB

(c) AP = 0

(d) AP = $\frac{1}{4}$ AB

22. Each of the letter pairs contained in the following paragraph refers to either a number or a line or a segment or a ray; in some cases a symbol which should appear above the letter pair is missing. Insert the missing symbols so that the resulting paragraph will be correct.

"AB + BC = AC. DB contains points A and C , but DB contains neither point A nor point C. A belongs to DB but C does not."

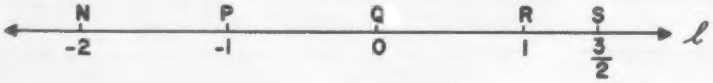
Draw a picture that illustrates your response.

23. R and S are points on ℓ with coordinates x_1 and x_2 respectively such that $x_2 > x_1$.
P is any point on RS with coordinate x. If $\frac{RP}{RS} = k$, match the following.
(Make the matching a one-to-one correspondence between the items in column A and those in column B.)

Column A	Column B
a) $k = 0$	1) RP = PS
b) $k = \frac{1}{2}$	2) P = R
c) $k = \frac{1}{3}$	3) P = S
d) $k \geq 0$	4) RP = $\frac{PS}{2}$
e) $k = 1$	5) P is in $\overline{RS}$
f) $0 \leq k \leq 1$	6) P is in $\overrightarrow{RS}$
g) $0 < k < 1$	7) P is in the interior of $\overline{RS}$
h) $k > 0$	8) P is in the interior of $\overrightarrow{RS}$

24. Let P , Q , X in coordinate system C have coordinates 1 , 0 , x respectively, and in coordinate system C' have coordinates 2 , 8 , x' respectively. Write an equation which expresses the relation between x and x'.

25. Given points with coordinates as indicated in the diagram.



State whether each of the following statements is true or false.

(a) $\overrightarrow{QR} = \overrightarrow{QS}$.

(b) $\overline{PQ}$ is a subset of $\overleftrightarrow{RS}$.

(c) Q and S belong to opposite rays which are contained in $\overleftrightarrow{NP}$ and have common endpoint R.

(d) $\overrightarrow{QS}$ is a subset of $\overrightarrow{PR}$.

(e) $\overrightarrow{QR}$ is a subset of $\overline{PS}$.

(f) The intersection of $\overrightarrow{PN}$ and $\overrightarrow{QR}$ is empty.

(g) The intersection of $\overrightarrow{PN}$ and $\overrightarrow{RQ}$ is $\overrightarrow{PN}$.

(h) The union of $\overrightarrow{QR}$ and $\overrightarrow{RQ}$ is $\overleftrightarrow{RQ}$.

(i) The union of $\overrightarrow{PN}$ and $\overrightarrow{PQ}$ is $\overleftrightarrow{PN}$.

(j) The intersection of $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ is $\overleftrightarrow{PQ}$.

26. In a given coordinate system on $\overleftrightarrow{MN}$, $\overline{MN}$ is the set of all points whose coordinates x satisfy $-2 \leq x \leq 6$. If it is known that the coordinate of M is less than the coordinate of N , answer the following questions:

(a) What is the coordinate of each endpoint of $\overline{MN}$?

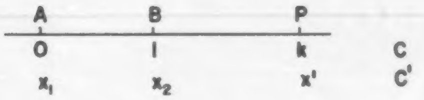
(b) What is the coordinate of the endpoint of $\overrightarrow{MN}$? of $\overrightarrow{NM}$?

(c) What is the coordinate of the midpoint of $\overline{MN}$?

27. A coordinate system C on line ℓ assigns coordinates 0, -1 , 4 to points Q , T , S respectively. Find the coordinate of S in coordinate system C' if Q and T have respective C' coordinates as follows:

(a) 0, 1
(b) -1, 0
(c) 1, -1
(d) 5, 2
(e) 1, 0

28.



Given the coordinate systems C and C' as shown in the diagram.

(a) Write an equation which expresses x' in terms of k , x_1, and x_2.

(b) If $x_1 = -3$ and $x_2 = 1$, find x' if
  (1) $k = 3$
  (2) $0 \leq k \leq 1$
  (3) $k = -2$
  (4) $k = \frac{1}{2}$

Part II.

Read each of the following statements carefully. If the statement is true as given, write true. If it is false, write a word or phrase which, if used in place of the underlined word or phrase, would change the statement to a true one.

1. Given two distinct real numbers, one of them is greater than the other. ________

2. Given two distinct real numbers, the first is greater than the second if and only if the difference obtained by subtracting the second from the first is a negative number.

____

3. A real number n is non-negative if and only if $n > 0$.

____

4. A number line is a convenient device for indicating the order and betweenness relations among numbers and "how far apart" two numbers are. ________

5. The number one may be the distance between any two distinct points in space. ________

6. The number zero is the measure of the distance from P to Q if the points P and Q are distinct. ________

7. The measure of the distance between two points P and Q, relative to a given unit-pair, is denoted by $\overline{PQ}$. ________

8. The number line of formal geometry corresponds to the general idea of a coordinate system. ________

9. One essential feature of a coordinate system is a matching of every point on a line with one and only one integer.

____

10. Any point of a line might be taken as the endpoint of some coordinate system. ________

11. The distance between two points whose coordinates are x and y is $x - y$. ________

12. Any two distinct points of a line can be taken as the origin and unit-point for at least three coordinate systems on the line. ________

13. If P, Q are endpoints of a segment, their coordinates must be 0, 1 respectively. ________

14. Two concurrent rays must also be opposite rays. ________

15. The intersection of two collinear rays may be a segment.

____

16. Given any segment, there exist exactly two coordinate systems which assign to its midpoint the number 0.5.

____

17. A segment has an interior. A ray also has an interior.

____

18. If P , Q , R have coordinates -2 , 4 , 1 respectively then Q bisects $\overline{PR}$. ________

19. The coordinate of every point of a ray is a non-negative number. ________

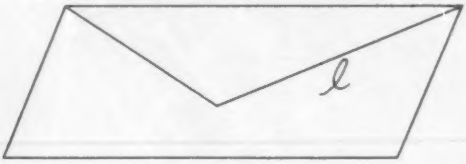
20. If the measure of the distance between the endpoints of segment $\overline{PQ}$ is 9 and the coordinate of P is 2 , then the coordinate of Q must be 11. ________

Chapter 4

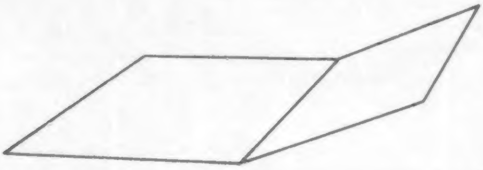
ANGLES

The pattern of development in this chapter on angles parallels to a certain extent that of the previous chapter. The individual sections by and large make convenient teaching units. The attitudes we expressed in this commentary concerning the formal proof of theorems in the chapter on distance and coordinate systems also apply to this chapter. The major teaching objective should be to help students to understand the statements of the definitions, postulates and theorems; teaching the details of the proofs is of secondary importance. Again it is usually appropriate to carry through numerical examples of the argument used in a general proof. Be careful, however, that students do not get the impression that this checking of numerical examples constitutes a proof. A consideration of specific examples will give the students not only a better understanding of the assertion of the theorem but also a good idea of why it is "reasonable," on the basis of our postulates and previous theorems.

The chapter begins with the notion of separation and the statement of two postulates concerning separation: One to the effect that a line in a plane separates or divides the plane into halfplanes; the other to the effect that a plane separates or divides space into halfspaces. In connection with these postulates is presented the notion of a convex set. (A set is convex if the segment joining any two points of it lies in the set.) We introduce this topic at this point to provide the means for defining the interior of an angle. Later it is used to define interior of triangles, polygons, etc. While this is a technical reason for introducing the postulates at this point, their general role in our description of the relations between points, lines and planes should not be overlooked. It may have been noted that we never use the adjective "straight" in connection with lines, nor do we use the adjective "flat" in connection with planes in our formal work. These new postulates together with our previous ones make it unnecessary to do this since they require in a very real sense that all planes are "flat". Postulate 8 (which says that if two points of a line are in a plane, then the line is in the plane) requires that planes be bent or curved or wiggly so that they "fit" the line through any two points of them. Postulate 14 , that a line in a plane separates or divides the plane into two convex sets, forces lines not to be "too wiggly." For instance, consider the "non-straight line" ℓ in the following diagram.



It certainly divides the plane into two parts, but these parts are not both convex. This same postulate, Postulate 14 , also has the effect of helping to insure that planes are "flat." For instance, as pointed out in the text, by this postulate a plane cannot be cylindrical, since a line along the length of a cylinder does not divide it into two parts. Postulate 15 , which says that a plane separates or divides space into two convex sets, completes the story by insuring that a plane is "flat." For instance, in the accompanying diagram the dihedral angle shown divides space into two parts but these parts are not both



convex and the dihedral angle is not "flat." From these remarks we hope that it is clear that these postulates collectively have the effect of insuring that all lines are "straight" and all planes are "flat." Since they do, these adjectives are unnecessary in our formal geometry.

After the introduction of the separation postulates in Section 4 - 2 , the text develops the concept of an angle as a set of points (namely, as the union of two concurrent rays rather than as a rotation), the idea of a ray-coordinate system in a plane and its connection with angle measurement, and related topics in Sections 4 - 3 through 4 - 7. In connection with angle measurement, you will not find in this chapter a discussion of the relations between different units of distance, such as the relation between radians and degree measure. We do not introduce this in our formal work; we use degree measure exclusively. Nonetheless, there is a strong parallelism between the developments in Chapters 3 and 4. We have tried in our writing to emphasize this parallelism. We feel that the analogies that exist between the treatments in the two chapters will not only make the understanding of our development of angles easier for students but also will reinforce the pattern of ideas used in developing the concept of distance and that of coordinates on a line. The chart that follows indicates these analogies.

The column on the left lists concepts and ideas relating the angles in the order of their occurrence in Chapter 4. The column on the right lists the corresponding ideas in Chapter 3. In teaching Chapter 4, you may wish to have the students work out individually such a list of analogies or you may wish to develop such a list on a chalkboard or bulletin board so that it can be referred to during the study of Chapter 4.

To show that the parallelism is not complete some ideas are listed in one column with a blank in the other. On occasion an idea clearly existed in one chapter but no technical term for it was defined. In these cases we have invented a term to add to the completeness of the list, putting our invention in quotation marks. We ask that you call your students' attention to the incompleteness of some of the phrasing and to the fact that we are speaking of analogies in some informal sense rather than in terms of a precise technicality (such as the duality that occurs in some geometries.)

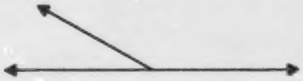
Analogies Between Concepts and Statements of Chapters 4 and 3.

Chapter 4	Chapter 3
1. Two concurrent, non-collinear rays (angle).	1. Two distinct points.
2. Measure of Angle, (Postulate 16)	2. Distance (between two points) (Postulate 10)

3. (See Commentary discussion of Section 4-5, of text).	3. Change of unit-pair (Postulate 13)
4. "Ray-Coordinates" in a plane Definition, Section 4-5 and Postulate 17	4. Coordinates on a line Definition, Section 3-5 and Postulate 12
5. Zero - ray Definition, Section 4-5	5. Origin Definition, Section 3-5
6. (See Commentary for Section 4-5)	6. Unit-point. Definition, Section 3-5
7. Theorem 4-3 (Angle Construction Theorem)	7. Theorem 3-8 (The Point Plotting Theorem.)
8. Three rays $\overrightarrow{VA}$, $\overrightarrow{VB}$, $\overrightarrow{VC}$ whose interiors are contained in a halfplane.	8. Three points A , B , C which are contained in a line.
9. $\overrightarrow{VB}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$ implies that in a ray-coordinate system, the ray-coordinate of $\overrightarrow{VB}$ is between the ray-coordinate of $\overrightarrow{VA}$ and $\overrightarrow{VC}$ (Definition of Betweenness for Rays, Section 4-6)	9. B is between A and C implies that in a coordinate system on the line containing A , B , and C , that the coordinate of B is between the coordinates of A and C. (Theorem 3-7, the Betweenness-Coordinates Theorem.)
10. Theorem 4-4 (The Betweenness-Angles Theorem)	10. Theorem 3-9 (The Betweenness-Distance Theorem.)
11. Midray (Definition, Section 4-6)	11. Midpoint (Definition, Section 3-7)
12. Theorem 4-5 (Existence and uniqueness of mid-ray.)	12. Theorem 3-3 (Existence and uniqueness of mid-point.)
13. Formula for coordinates of mid-ray (End of Section 4-6) (Note restriction on b - a)	13. Formula for coordinates of mid-point (Section 3-10) (Note that there is no restriction on b - a)

14. Interior of $\angle AVC$ = (set of points B, such that $\overrightarrow{VB}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$; Postulate 18, Interior of an Angle Postulate and Definition of Interior of an Angle, Section 4-7.)	14. Interior of segment $\overline{AC}$ = Set of points, B , between A and C ; (See Section 3-7.)
15. Interior of $\angle AVC$ (The intersection of the half-plane determined by $\overleftrightarrow{AV}$ and C and the halfplane determined by $\overleftrightarrow{CV}$ and A. We note that the edges of the halfplanes do not lie in the halfplanes. Postulate 18. Interior of an Angle Postulate and Definition of Interior of an Angle, Section 4-7.)	15. Interior of segment $\overline{AC}$. (The intersection of the interior of $\overrightarrow{AC}$ and the interior of $\overrightarrow{CA}$. We note the endpoints of the rays do not lie in the interiors of the rays. Related to (g) of Example 1, Section 3-6.)
16. "Wedge AVC" (The union of $\angle AVC$ and the interior, not defined in text. "Wedge AVC" could be called "Fan AVC" if it is thought of as a family of rays.)	16. Segment $\overline{AC}$ (Union of two endpoints and interior; definitions, Section 3-6 and 3-7.)
17. Theorem 4-6. The interior of an angle is a convex set.	17. The interior of a segment is a convex set. (It is the intersection of two halflines, hence convex by remark on page 135 and Theorem 4-1.)
18. Exterior of an angle (Definition, Section 4-7)	18. Not defined (Analogous to the points of a line not in a given segment of the line.)
19. Congruent angles (Definition, Section 4-8.)	19. Congruent segments (Definition, Section 3-11.)

In section 4-8 , the idea of adjacent angles, that is, of two coplanar angles with a side in common and with disjoint interiors, is introduced. The special case in which the two sides not in common are collinear yields the notion of a



linear pair. The special case in which the two angles of a linear pair are congruent yields the notion of a right angle. This way of introducing right angles is chosen because it indicates that right angles as such are intrinsic to Euclidean Geometry. It is of course natural to discuss perpendicularity along with right angles.

Section 4-9 , deals with supplementary and complementary angles; Section 4-10 concerns vertical angles.

The chapter concludes with Sections 4-11 and 4-12 which introduce triangles, quadrilaterals, and polygons and related topics. Triangles and triangle congruences appear frequently in the next chapter. In connection with the work of Chapter 5 , students will start to write many proofs of their own.

134 Crudely speaking a set is convex if it has no indentations. It is not appropriate to say that it is "rounded" since a triangle together with its interior is convex and has "sharp corners"; that is, is not "rounded."

136 Notice the union of two convex sets is not necessarily convex. For instance, every line is a convex set, but the union of two distinct lines is clearly not a convex set.

138 The reason for not hyphenating or writing as separate words, words such as halfplane, is to make it unequivocally clear in a definition such as that for halfline that the entire phrase "halfline" is being defined and that neither one of the two parts is being defined.

138 From the Plane Separation Postulate we can deduce the following result: If points A , B , C of a plane m are such that A and B are on the same side of a line ℓ in m and B and C are on the same side of ℓ, then A and C

are on the same side of ℓ. This can be established by observing that each of the points A, B, and C is in exactly one of the two halfplanes, H_1 and H_2 with edge ℓ. By hypothesis, A and B are in the same halfplane, as also are B and C. If A and B are both in H_1, then B and C must be in H_1 and A and C must be in H_1. If A and B are in H_2, then B and C must be in H_2 and A and C must be in H_2. In either case, A and C are on the same side of ℓ.

We can also deduce the following result: If A, B, C, and ℓ are coplanar and if ℓ intersects $\overline{AB}$ in an interior point but does not contain C then ℓ must also intersect the interior of one of the segments $\overline{AC}$ and $\overline{BC}$. This can be established by observing that, by hypothesis, A and B are on opposite sides of ℓ and that C is not on ℓ. Consequently C is either on the same side of ℓ as A or on the same side of ℓ as B. If it is on the same side of ℓ as A, then ℓ intersects $\overline{BC}$ at an interior point. If it is on the same side of ℓ as B, then ℓ intersects $\overline{AC}$ at an interior point. This completes the proof. Sometimes this theorem is expressed by saying that a line in the plane of a triangle and intersecting one side of the triangle in an interior point must intersect one of the other two sides in an interior point provided it does not contain a vertex.

140 From Postulate 15, we can prove in much the same fashion as indicated in our comments on the Plane Separation Postulate that:

(1) If points A, B, C are such that A and B are in the same halfspace determined by a plane m and such that B and C are in the same halfspace determined by m, then A and C are in the same halfspace determined by m.

(2) If A, B, C are such that plane m intersects $\overline{AB}$ in an interior point and does not contain C, then m must also intersect the interior of one of the segments $\overline{AC}$ and $\overline{BC}$.

140 When we say a plane determines exactly two halfspaces we mean in contrast to our earlier usage of "determine" not that

the plane contains the halfspaces but that it specifies or points them out in an unambiguous manner.

144 Principally we are concerned with angles of triangles. No triangle has an angle whose measure is zero or which is a straight angle. Indeed, the measure of any angle of a triangle is between 0 and 180. The concept of a linear pair of angles in introduced later (Section 4-8) to take account of the fact that the exterior angle of a triangle and its adjacent interior angle form a "straight angle" or have measures that add up to 180. For these reasons the zero angle and the straight angle are unnecessary, and we can confine our attention to angles whose measures are between 0 and 180. Incidentally, Euclid avoided the use of "straight angle", too.

We merely make a distinction between what is needed in elementary synthetic and analytic geometry and what is needed in trigonometry and analysis. Our definition is satisfactory and the simplest for our needs. As the student advances in mathematics extensions of the definition of angular measure will be made and then definitions of angle will be introduced as the need arises.

145 As indicated in the chart of analogies with Chapter 3, angle and segment correspond. Thus, corresponding to the role of the two endpoints determining the segment, we have two rays determining an angle. We recognize, of course, that the statement that two rays determine an angle needs the addition of the word concurrent to describe the rays. The analogue of this with the points determining the segments is that the points are collinear (a trivial statement for the points, but the corresponding requirement of concurrency for the rays is not trivial.) Again, it is important that the rays be noncollinear for if they are collinear, they do not form an angle. Analogously, it is important that the points be distinct or they do not determine a segment.

146 In the use of the word determine, when we speak of the angle determined by the two noncollinear segments we, of course, mean the union of the unique pair of concurrent rays that contains the union of the segments as a subset.

154 Contrast and call attention to the difference between an angle and the measure of an angle. Stress that an angle is a set of points while its measure is a number. Such a distinction between the point set and the number associated with it is not made in most textbooks, the word "angle" being used for both.

Although we will restrict ourselves to degree measure exclusively in our postulational treatment of angles, other measures are possible (as we have already suggested) and a development of angle measurement paralleling our development of the measurement of distance in Chapter 3 can be given. To begin with, Postulate 16 can be stated in more general terms by replacing the number 180 by any other positive number, say R. This raises the following question: "How are angle measures based upon different values of R related?" To provide the desired answer to this question, another postulate, analogous to Postulate 13 , must be introduced. This would say, in effect, that if $m\angle ABC$ (with respect to R_1) and $m\angle ABC$ (with respect to R_2) are the measures of any angle, ABC, in terms of two different values of R , then

$$m\angle ABC \text{ (with respect to } R_1) = \frac{R_1}{R_2}\, m\angle ABC \text{ (with respect to } R_2)$$

In particular, this provides us with the familiar relation connecting radian measure (for which $R = \pi$) and degree measure (for which $R = 180$), namely,

$$m_{rad}\angle ABC = \frac{\pi}{180}\, m_{deg}\angle ABC$$

On the basis of the new postulate, a theorem analogous to Theorem 3-4 can be framed. This result and others assure us that such things as betweenness relations for rays and the determination of the ray which forms, with a given angle, two angles whose measures have a prescribed ratio are independent of the unit which is used.

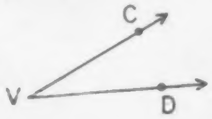
The teacher might feel a postulational development such as that of our text leaves a gap in the development. This is not the case. The postulates that we assume do give us Euclidean geometry. The lack is solely an inadequacy in the technical language that we have chosen. We can only speak of degrees.

Since the past experience of the student has probably been largely limited to degrees, it is unlikely that he will question this language deficiency. Should he challenge it the odds are that he is a student of such a caliber that you could let him study for himself the discussion in the Commentary above - so that he can see that we can parallel our treatment of distance and obtain other measures for angles.

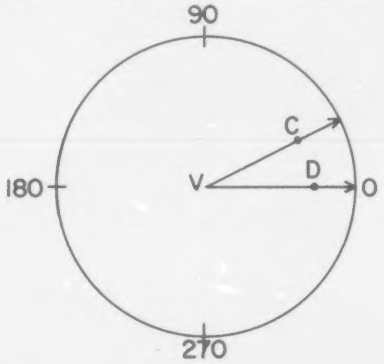
159 One aspect of our restriction to degree measure is that although the zero-ray is clearly analogous to the origin in a coordinate system on a line, there is no real role played by what might be called the unit-ray. (Indeed, in some systems of angle measure, for instance, in going from 0 to .3 instead of 0 to 360 , there might be no ray corresponding to the number one. See the chart of analogies.)

159-160 Note that in using the Protractor Postulate in our geometric development, it is always necessary to specify how the "protractor is to be placed" (i.e. the ray $\overrightarrow{VA}$ and $\overrightarrow{VB}$ must be specified) before the other assertions of the postulate can be used.

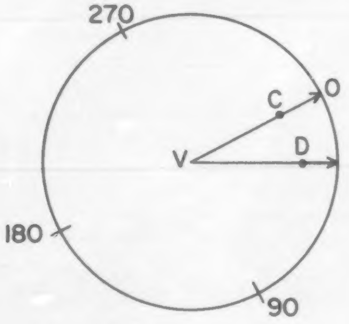
Consider the rays $\overrightarrow{VC}$ and $\overrightarrow{VD}$ as shown below:



If $\overrightarrow{VD}$ is to play the role of $\overrightarrow{VA}$, and $\overrightarrow{VC}$ that of $\overrightarrow{VB}$, then the circular protractor is placed as shown below:



If, however, $\overrightarrow{VC}$ is to play the role of $\overrightarrow{VA}$ and $\overrightarrow{VD}$ that of $\overrightarrow{VB}$, then the circular protractor is placed as shown below:



Diagrams such as those above, if drawn on the board, may help the student to vizualize the significance of the placement aspect of the postulate as well as the other parts of the statement of the postulate. Similar diagrams may be useful in explaining the significance of the definitions of a ray-coordinate system, Theorem 4-3, as well as in connection with later work.

160 Notice that in Theorem 4-3 a part of the conclusion is that there is a unique ray $\overrightarrow{VR}$ whose interior, not the ray itself, is in the halfplane, the endpoint V is not in the halfplane.

A proof of Theorem 4-3 may be given as follows:

In plane m let H be a halfplane whose edge contains the ray $\overrightarrow{VA}$. In H is a point B. By Postulate 17, there is a unique ray-coordinate system in m relative to V such that $\overrightarrow{VA}$ corresponds to 0 and every ray $\overrightarrow{VX}$ with X in H corresponds to a number less than 180. Since r is a number between 0 and 180 there is in this ray-coordinate system, a ray $\overrightarrow{VR}$ such that R is in H and $\overrightarrow{VR}$ corresponds to r. Furthermore $m\angle AVR = r$. We have now shown that there exists a ray $\overrightarrow{VR}$ as required; it remains to show that it is unique. Suppose $\overrightarrow{VS}$ is any ray such that S is in H and $m\angle AVS = r$. Let the coordinate of $\overrightarrow{VS}$ be s. By the Protractor Postulate, $s < 180$ and hence $m\angle AVS = s - 0 = s$.

Thus $s = r$. But then, since a ray-coordinate system is a one-to-one correspondence between rays and numbers, $\overrightarrow{VS} = \overrightarrow{VR}$. Hence $\overrightarrow{VR}$ is unique.

164-165 Our remarks are not intended to be a defense of our particular definition of betweenness for rays. They are intended merely to show that our definition is reasonable. We freely admit that other definitions are not only possible but might be equally convenient in the development of geometry. Classroom discussion of diagrams such as those in Figure (a) may lead to the following:

In (1), students will probably agree that $\overrightarrow{VQ}$ is between $\overrightarrow{VR}$ and $\overrightarrow{VP}$. However, if (a) was construed as three dimensional by taking V , R , Q , P as the vertices of a pyramid, then there is probably equally good reason to say $\overrightarrow{VR}$ is between $\overrightarrow{VQ}$ and $\overrightarrow{VP}$ as to say $\overrightarrow{VQ}$ is between $\overrightarrow{VR}$ and $\overrightarrow{VP}$, or as to say $\overrightarrow{VP}$ is between $\overrightarrow{VR}$ and $\overrightarrow{VQ}$. To avoid such ambiguity we have restricted the notion of betweenness for rays to coplanar rays.

In (2), although one can say $\overrightarrow{VP}$ is between $\overrightarrow{VR}$ and $\overrightarrow{VQ}$, it is equally reasonable to say $\overrightarrow{VR}$ is between $\overrightarrow{VQ}$ and $\overrightarrow{VP}$ or to say $\overrightarrow{VQ}$ is between $\overrightarrow{VR}$ and $\overrightarrow{VP}$. From this ambiguity, we get the idea that if the rays "go all around the compass" then there is difficulty with the betweenness notion. Hence, we arbitrarily restrict betweenness for rays to those that lie in or on the edge of a halfplane and such that no two of them are collinear.

The same remarks as those for (1) can be made for (3).

In (4) it is perfectly reasonable to say $\overrightarrow{VQ}$ is between $\overrightarrow{VR}$ and $\overrightarrow{VP}$. However, this is not in accord with our definition. In our definition we would require that the interior of the ray $\overrightarrow{VQ}$ is in the interior of the angle formed by the other two rays $\overrightarrow{VR}$ and $\overrightarrow{VP}$. However, because of the ambiguity about the idea of interior for a "straight angle", we do not in our development consider the union of $\overrightarrow{VR}$ and $\overrightarrow{VP}$ to be an angle.

Again in (5) it is probably reasonable to say $\overrightarrow{UQ}$ is between $\overrightarrow{VR}$ and $\overrightarrow{VP}$. However, this is not in accord with the definition we have chosen. In our definition, as a matter of technical convenience, we have required that the three rays be concurrent.

Note also the entries on betweenness in the chart of analogies.

165 We have not brought out to the student that if $\overrightarrow{VB}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$ then we need a theorem to establish the fact that $\overrightarrow{VB}$ is also between $\overrightarrow{VC}$ and $\overrightarrow{VA}$. In our opinion this should not be brought out in class by the teacher nor would we feel it appropriate to raise the question with most secondary school students. We are content to let students exchange freely the role of $\overrightarrow{VA}$ and $\overrightarrow{VC}$ as indicated above. The needed theorem can be established as follows. Suppose $\overrightarrow{VB}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$ and that in a ray-coordinate system based on V in the plane the ray coordinates of $\overrightarrow{VA}$, $\overrightarrow{VB}$, $\overrightarrow{VC}$ are, respectively, 0, b, c where $0 < b < c < 180$. Note that

$$m\angle AVC = c,\ m\angle AVB = b,\ m\angle BVC = c - b.$$

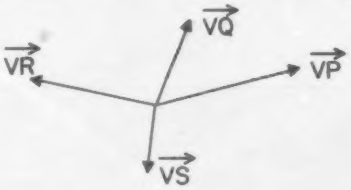
Now consider a ray-coordinate system based on V such that $\overrightarrow{VC}$ is the zero-ray and the ray-coordinate of $\overrightarrow{Va}$ is less then 180. The new ray-coordinate of $\overrightarrow{VA}$ must then be c since $m\angle AVC = c$. Let us suppose the new ray-coordinate of $\overrightarrow{VB}$ is x. We proceed by elimination of cases to show that $x = c - b$. Suppose $x > c$. Then $m\angle BVC = x$ or $360 - x$ depending on the size of x. Hence, since we know $m\angle BVC = c - b$, either $x = c - b$ or $360 - x = c - b$. Since $x > c$, it is impossible that $x = c - b$. Hence if $x > c$, then the other possibility holds, namely, $x = 360 - c + b$. Now $m\ BVA = b$ and by calculation in the new ray-coordinate system $m\ BVA = x - c$ or $360 - (x - c)$. That is, $b = x - c$ or $b = 360 - (x - c)$. In the first case, substituting $x = 360 - c + b$, we obtain $b = 360 - c + b - c$ which implies $c = 180$ which is impossible. In the second, substituting $x = 360 - c + b$, we obtain $b = 360 - (360 - c + b - c)$ which implies $b = c$ which is also impossible. Consequently, we must have $x < c$.

But then, since $m\angle BVC = x$ and $m\angle BVC = c - b$, we conclude that $x = c - b$. Now $0 < c - b < c < 180$, therefore $\overrightarrow{VB}$ is by definition between $\overrightarrow{VC}$ and $\overrightarrow{VA}$.

166-
167 The proof of Theorem 4-4 is left as Problem 4 in Problem Set 4-6. This problem is in the form of an almost completed proof with a few blanks to be filled in by the student. The proof of Theorem 4-5 is left as Problem 5 in Problem Set 4-6. This problem also is of the fill-in type.

Note the analogies between these theorems and those in Chapter 3. See the chart of analogies, and the preface to Problems 4 and 5 of Problem Set 4-6.

167 Notice that in defining the meaning of the phrase "concurrent in that order" it was necessary to say that each of $\overrightarrow{VQ}$ and $\overrightarrow{VR}$ is between $\overrightarrow{VP}$ and $\overrightarrow{VS}$ so as to avoid the situation in the diagram below, where $\overrightarrow{VQ}$ is between $\overrightarrow{VP}$ and $\overrightarrow{VR}$ and $\overrightarrow{VR}$ is between $\overrightarrow{VQ}$ and $\overrightarrow{VS}$ but neither $\overrightarrow{VQ}$ nor $\overrightarrow{VR}$ is between $\overrightarrow{VP}$ and $\overrightarrow{VS}$. Note the analogous situation in defining the phrase "collinear in that order" (see page 91 of the Text). Indeed, P , Q , R , S are collinear in that order, if Q is between P and R and R is between Q and S. For we can then deduce that each of Q and R is between P and S.



167 Note the analogy between the formula for the ray-coordinate of the midray and that for the midpoint of a segment. The formula for the midpoint of a segment is in Section 3-10. Note also that in order to use the formula for angles one must first check that the smaller ray-coordinate subtracted from the larger is less than 180. There is no similar restriction in computing the coordinate of the midpoint of a segment. For a derivation of the midray formula see the

answers to Problem 9 of Problem Set 4-6. The formula can be motivated by numerical experiments.

Sometimes it may be convenient to speak of $\overline{AB}$ or $\overleftrightarrow{AB}$ as a bisector if $\overrightarrow{AB}$ is a midray. If such terminology is to be used in the classroom, the teacher should give the students a formal definition.

174 The mathematician-teachers on the GW writing team have not hesitated to include the statement of Postulate 18 as a postulate rather than a theorem. In our capacity as teachers we recognize clearly that the extent to which betweenness proofs should be "pushed" with secondary school students is limited. We also feel that to state it as a theorem without proof is not as desirable as stating it frankly as a postulate. Before showing how it could be established as a theorem, we remark that assuming S to be the same as R and I is equivalent essentially to assuming the parallel postulate. We will say more about this after showing that $R = I$ and that S is contained in R.

To show that $R = I$ we must show both that R is contained in I and that I is contained in R. We first show that R is contained in I. Suppose that P is in R. Then P is on a ray $\overrightarrow{VP}$ between $\overrightarrow{VA}$ and $\overrightarrow{VB}$. Apply the Protractor Postulate so that $\overrightarrow{VA}$ is the zero-ray and $\overrightarrow{VB}$ corresponds to a number less than 180. Then by the definition of betweenness for rays the ray-coordinate of $\overrightarrow{VP}$ is between 0 and 180. Hence, by the Protractor Postulate, P is on the same side of $\overleftrightarrow{VA}$ as B. Similarly, using the Protractor Postulate to obtain a ray-coordinate system such that $\overrightarrow{VB}$ is the zero-ray and $\overrightarrow{VA}$ has a ray-coordinate between 0 and 180, we argue using the definition of betweenness for rays and the Protractor Postulate that P is on the same side of $\overleftrightarrow{VB}$ as A. Since we have shown that P is in the half plane with edge $\overleftrightarrow{VA}$ and containing B as well as the half plane with edge $\overleftrightarrow{VB}$ containing A, we have established that R is contained in I.

We now show that I is contained in R to complete the proof that $R = I$. Suppose that P is in I. Apply the Protractor Postulate so that $\overrightarrow{VA}$ is the zero-ray and $\overrightarrow{VB}$ corresponds to a number less than 180. We know that the ray-coordinate p of $\overrightarrow{VP}$ is between 0 and 180, since P is in the halfplane determined by $\overleftrightarrow{VA}$ and B. Where the ray-coordinate of $\overrightarrow{VB}$ is b, there are two possibilities:

$$(1) \quad b < p ,$$
$$(2) \quad p < b .$$

We will show that possibility (1) cannot hold and therefore that (2) must hold. In case (1) holds, we know that

$$m\angle AVB = b$$
$$m\angle AVP = p$$
$$m\angle BVP = p - b$$

Applying the Protractor Postulate again so that $\overrightarrow{VB}$ is the zero-ray and $\overrightarrow{VA}$ corresponds to a number less than 180, we know that the new ray-coordinate of $\overrightarrow{VP}$ is between 0 and 180 since P is in the halfplane determined by $\overleftrightarrow{VB}$ and A. Hence it must be $p - b$ since $m\angle BVP = p - b$. The new ray-coordinate of $\overrightarrow{VA}$ must be b since it is between 0 and 180 and $m\angle AVB = b$. Since the ray-coordinate of $\overrightarrow{VP}$ is $p - b$ and that of $\overrightarrow{VA}$ is b, $m\angle AVP$ as computed in the new ray-coordinate system is one of two possibilities. Since each of the numbers b and $p - b$ is less than 180, these possibilities are

$$(p - b) - b = p - 2b$$
$$b - (p - b) = 2b - p,$$

But we know $m\angle AVP = p$ that $b \neq 0$ and that $b \neq p$; hence neither of these possibilities can hold. Consequently our original possibility (1) does not hold. Therefore, $p < b$, or in other words, $\overrightarrow{VP}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VB}$. Hence I is contained in R. Since we have established both that I is contained in R and that R is contained in I, we have established that $R = I$.

We next prove that S is contained in I. Let P be a point of S. Since P is a point of S, there exists points C and D such that C is in the interior of $\overrightarrow{VA}$, D is in the interior of $\overrightarrow{VA}$. Since $\overleftrightarrow{CD}$ intersects $\overleftrightarrow{VA}$ at C only, the segment $\overline{CD}$ except for C lies in one of the halfplanes, determined by $\overleftrightarrow{VA}$. Further, P is the same halfplane determined by $\overleftrightarrow{VA}$ with D. Similarly P is the same halfplane determined by $\overleftrightarrow{VB}$ with C. Hence P is in the intersection these halfplanes; that is, P is in I.

To see that the statement, S = I, implies the Parallel Postulate we ask you to consider Theorem 1, in Part I of the Talk, Introduction to Non-Euclidean Geometry, which appears in Volume II of this Commentary. From this theorem we see that if there are two lines parallel to given line ℓ through a point V not on ℓ, then there is an angle $\angle AVB$ contained in the union of the lines such that ℓ is in the interior of $\angle AVB$. We observe that $\angle AVB$ lies entirely on one side of ℓ. Now any segment containing a point P of ℓ and with endpoints in the sides $\angle AVB$ is such that the endpoints of the segment must be on opposites sides of ℓ. This contradicts the fact that $\angle AVB$ lies entirely on one side of ℓ. Consequently if every point of the interior of any angle is on a segment joining the sides of the angle, it is impossible to have two distinct lines through a point parallel to another line. Thus, S = I implies the Parallel Postulate.

175-176 From the diagram on page 175, it is probably clear to you that Theorem 4-7 is preparatory to the usual proof of the Exterior Angle Theorem which occurs at the end of Chapter 5 in our text. A complete proof of the Exterior Angle Theorem requires among other things a proof that $\overrightarrow{CF}$ is between $\overrightarrow{CB}$ and $\overrightarrow{CG}$. This is the purpose of Theorem 4-7.

The students should reproduce the diagram on page 175 so that they can have it in sight as they read the theorem and its proof. As an exercise in the use of available formal language which does not include thw word "triangle" or "exterior angle of a triangle" at this stage, it is useful to ask students to describe in words the configuration of the diagram. This exercise should serve to make clear the choice of language in the theorem. It is also useful to ask the students to check their description in words by seeing if they can draw a figure which satisfies their description but is unlike the original drawing. It is important that all students understand the statement of this theorem; for many classes it may be appropriate to omit a discussion of the proof in class and to tell the students that they may use the theorem in later proofs as needed.

179 In the definition of a linear pair it is not necessary to state that the angles are coplanar, since two of the concurrent rays are opposite rays. The concept of linear pair is the means by which we avoid the concept of right angle. Note also that the ray common to the two angles is not between the other two rays.

180 In the definition of adjacent angles it is necessary to state that the two angles are coplanar. This is in contrast to the situation with the linear pair.

Note also that just because three rays form adjacent angles, it is not necessarily true that the ray common to the two angles is between the other two. After all, the three rays are not necessarily contained in any halfplane and its edge.

194 Since the definition of a pair of vertical angles involves opposite rays, it is clear that the two angles of a pair of vertical angles are coplanar.

195 In the statement of Theorem 4-18, each line is determined in the sense that it is the unique line that contains the given set. Each pair of vertical angles is determined in the sense that the union of the angles contains the given sets and this union is the unique set which is the union of two distinct lines containing the given sets.

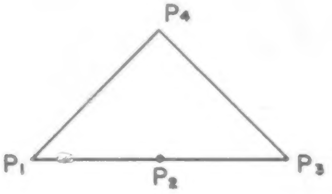
The formal proof of the theorem, interpreting the use of the word determine as above, consists in observing that the point of intersection, V, of the two distinct lines, separates each of the lines into the interior of two opposite rays. There are exactly two ways of using these two pairs of opposite rays to form a pair of vertical angles. The two pairs of vertical angles thus formed are the required pairs of vertical angles.

201 In connection with the use of the word determine in the definition of a triangle, when two points determine a segment, they are invariably taken to be the end points of the segment.

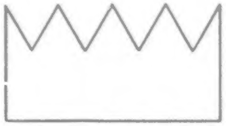
209 It might be useful to ask students the following questions in connection with the definition of a polygon. If $n = 3$, is it necessary to specify that the vertices are coplanar? Obviously, the answer is no. Another question might be the following:

> Could (2) be replaced by the requirement that $P_1, P_2, \ldots, P_n$ be noncollinear?

In answering such a question they might consider the following figure in which P_1, P_2, P_3, P_4, are noncollinear.



In this figure would it be natural to call P_2 a vertex? It could also be pointed that three or more vertices could be collinear as suggested in the following figure:



211 Notice that there are no triangles that are not convex polygons. Note also that this is not established by Theorem 4-22; for the "rest of the polygon" in this definition means the set of points of the sides of the polygon that are not also in the side which is in the edge of the halfplane. Notice also that while there are such things as convex polygons, there is no polygon which is a convex set.

Illustrative Test Items for Chapter 4

I. Read each statement carefully. If it is true as written, write "true." If not, write a word or phrase which could be substituted for the underlined word or phrase to make the statement true.

1. If *one* segment whose endpoints are in a point set S is entirely contained in S, then S is a convex set. ____________

2. If two convex sets intersect, their *union* is a convex set. ____________

3. The relation of a halfplane to a plane and the relation of a halfspace to a space are like the relation of a *ray* to a line. ____________

4. In this text we think about an angle primarily as a *rotation* of two concurrent noncollinear rays. ____________

5. We often denote the angle formed by the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ as *$\angle ABC$*. ____________

6. If segments $\overline{PQ}$ and $\overline{PR}$ are noncollinear then the union of *$\overrightarrow{PQ}$ and $\overrightarrow{PR}$* is $\angle QPR$. ____________

7. The measure of an angle formed by two opposite rays is *not defined* in our geometry. ____________

8. To find the measure of an angle we first compute the difference in the *measures* of the rays which are the sides of the angle, subtracting the smaller from the larger so as to get a positive number. ____________

9. The degree *is the only* unit of angle measure used in the formal geometry of this text. ____________

10. There is a correspondence which associates with each angle in space a unique *positive integer* less than 180. ____________

11. The measure of an angle is a positive number *less than 180*. ____________

12. The measure of the interior of an angle is undefined in our text. ____________

13. The ray-coordinate of the angle bisector of an angle is the average of the ray-coordinates of the sides of the angle provided the smaller ray-coordinate subtracted from the larger is less than 180. ____________

14. If one ray is between two others, the three rays must intersect. ____________

15. The exterior of an angle is a convex set. ____________

16. If the sum of the measures of two adjacent angles is 180 the two angles form a linear pair. ____________

17. If two coplanar angles are adjacent they have a common side and their interiors do not intersect. ____________

18. The sum of the measure of two distinct right angles is 90. ____________

19. In order for two lines to be perpendicular they must contain two rays whose union is a right angle. ____________

20. Two angles whose measures are the same are equal. ____________

21. Two acute angles cannot be supplements of each other. ____________

22. If one angle is acute and another angle is obtuse they cannot be a pair of vertical angles. ____________

23. $\angle ABC$ of $\triangle ABC$ is the union of $\overline{AB}$ and $\overline{AC}$. ____________

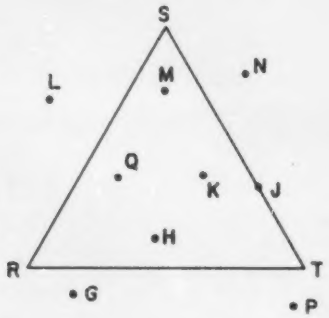
24. A triangle has three diagonals. ____________

25. The relation of side and vertex to angle is like the relation of face and edge to dihedral angle. ____________

II. Select the correct answer for each of the following problems:

Problems 1 - 2 refer to Figure (1); Problems 3 - 5 refer to Figure (2); and Problems 6 - 8 refer to Figure (3). All points in each figure are coplanar and points which appear to be collinear are collinear.

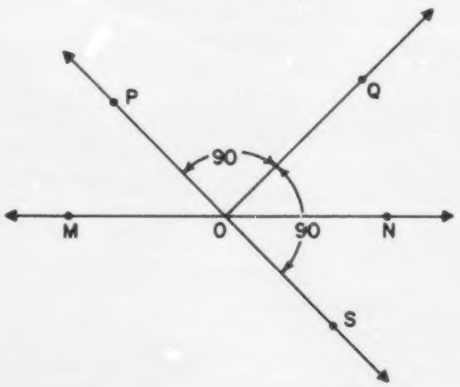
Figure (1)



1. Which of the following points is not in the interior of any angle of the triangle?
   (a) L (c) H (e) none of these
   (b) P (d) N

2. Which of the following points is in the exterior of ΔRST?
   (a) G (c) H (e) none of these
   (b) R (d) J

Figure (2)



3. $\angle MOP$ and $\angle NOS$ are:

(a) supplementary angles	(d) vertical angles
(b) adjacent angles	(e) none of these
(c) complementary angles	

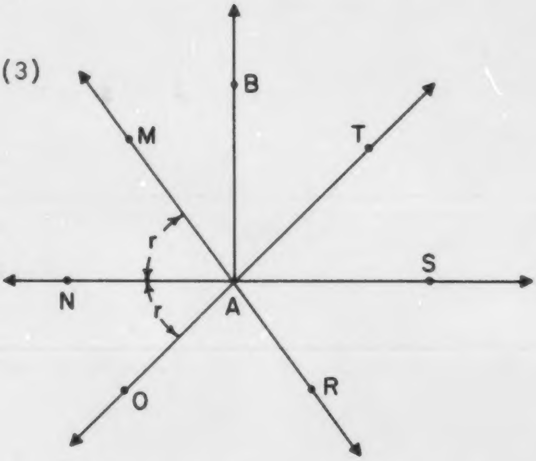
4. $\angle QON$ and $\angle NOS$ are:

(a) supplementary angles	(d) vertical angles
(b) right angles	(e) none of these
(c) complementary angles	

5. $\angle QOS$ is:

(a) a right angle	(d) a vertical angle
(b) an obtuse angle	(e) none of these
(c) an acute angle	

Figure (3)



6. If $\overrightarrow{AB} \perp \overleftrightarrow{NS}$, then $\angle NAB \cong \angle SAB$ because:
   (a) they are both acute angles.
   (b) they are complements of congruent angles.
   (c) they are both right angles.
   (d) they are vertical angles.
   (e) none of these.

7. $m\angle MAT$ equals:

(a) 180	(c) 180 - 2r	(e) none of these
(b) 2r	(d) 180 - r	

8. $\angle MAN$ is congruent to:

(a) $\angle BAT$	(c) $\angle BAM$	(e) none of these
(b) $\angle TAS$	(d) $\angle BAN$	

III. 1. Below are a number of statements and phrases in one column and a list of words or expressions in the other. Complete each item in the left solumn so as to make a true statement by selecting the proper word or expression from the right-hand column.

(a) An angle with measure less than 90 is __________.

(b) If $m\angle B$ is 60, the measure of a supplement of $\angle B$ is __________.

(c) The measure of a right angle is __________.

(d) If $\angle ABC$ is a right angle, then rays $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are __________.

(e) Angles with the same measure are __________.

(f) If $m\angle\alpha$ is 60, the measure of a complement of $\angle\alpha$ is __________.

(g) If the sum of the measures of two angles is 90, the angles are __________.

(h) An angle with a measure of more than 90 is __________.

(i) A supplement of a right angle has a measure of __________.

(j) Complements of congruent angles are __________.

(k) If $m\angle ABC + m\angle RST = 90$, then $\angle ABC$ is a __________ of $\angle RST$.

(l) A supplement of an acute angle is __________.

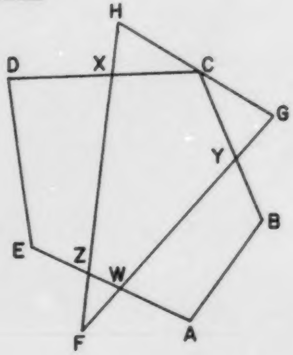
perpendicular
obtuse
right
90
acute
120
triangle
complement
congruent
30
n
complementary
supplementary

(m) If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays and $\overrightarrow{AE}$ is situated so that $m\angle CAE = m\angle BAE$, $\angle CAE$ is a ___________ angle.

(n) If the measure of an angle is twice that of one of its supplements, then it is __________.

(o) The measure of an angle whose measure is half that of one of its complements is ___________.

2. Given the triangle and pentagon in the figure. Describe the intersection of the following sets of points:



(a) $\triangle HFG$ and polygon ABCDE
(b) $\triangle FHG$ and $\overline{DE}$
(c) $\overline{HG}$ and $\angle DCB$
(d) $\angle HFG$ and polygon ABCDE
(e) Interior of $\triangle HFG$ and interior of polygon ABCDE

3. (a) If one of a pair of vertical angles has a measure of x, express the measures of the other three angles formed by the two intersecting lines in terms of x.

(b) If three rays have a common endpoint and two of them are opposite rays, what is the sum of the measures of the two angles formed by these rays?

(c) H is a point in the interior of $\angle RST$. $m\angle HST = 10$ and $m\angle RST = 30$. What is the value of $m\angle HST$?

(d) If two congruent angles are supplementary, what kind of angles are they?

(e) If each angle in one pair of vertical angles has measure 1, what is the measure of each angle of the other pair of vertical angles formed by the intersecting lines?

(f) If the difference between the measures of two complementary angles is 8, what is the measure of each angle?

4. Assume $\angle\beta$ is a supplement of $\angle\alpha$. Find $m\angle\beta$, if:

(a) $m\angle\alpha = 30$

(b) $m\angle\alpha = 120$

(c) $m\angle\alpha = n$

(d) $m\angle\alpha = 45 - n$

5. Suppose $\angle\beta$ is a complement of $\angle\alpha$. Find $m\angle\beta$ if:

(a) $m\angle\alpha = 38$

(b) $m\angle\alpha = 49$

(c) $m\angle\alpha = n$

(d) $m\angle\alpha = n + 25$

6. $\overrightarrow{XA}$ and $\overrightarrow{XB}$ are opposite rays on the edge of a halfplane $\mathcal{H}$. S and R are points of $\mathcal{H}$ such that $m\angle RXB = 35$ and $m\angle RXS = 90$. Make a sketch and answer the following:

(a) Name a pair of perpendicular rays in the sketch, if any occur.

(b) Name a pair of complementary angles in the sketch, if any occur.

(c) Name a pair of vertical angles in the figure, if any occur.

(d) Name two pairs of supplementary angles in the sketch, if two pairs occur.

(e) Name two acute angles in the sketch, if any occur.

(f) Name two obtuse angles in the sketch, if any occur.

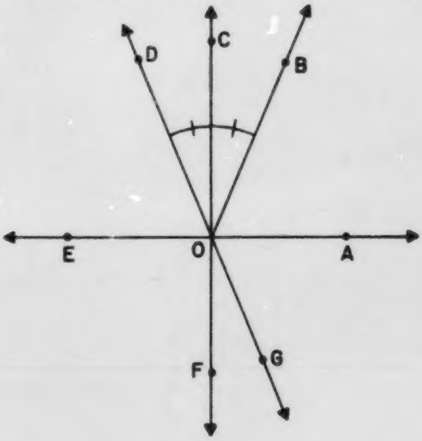
7. (a) Given coplanar rays $\overrightarrow{MA}$, $\overrightarrow{MB}$, $\overrightarrow{MC}$ to which a correspondence as described by the Protractor Postulate assigns real numbers a, b, c, respectively. Express in terms of a, b, and c,

   (1) $m\angle AMB$ if $a < c < b < 180$.

   (2) $m\angle CMP$ if $a < b < c < 180$ and $\overrightarrow{MP}$ is a midray of $\angle CMB$.

   (b) In part (a)2 above, find the real number, p, assigned by the correspondence to $\overrightarrow{MP}$.

8. Given three concurrent lines as in the figure with ray $\overrightarrow{OB}$ such that $\overleftrightarrow{AE} \perp \overleftrightarrow{CF}$ and $\angle DOC \cong \angle COB$. For each of the congruences below, state the theorem which justifies it.



   (a) $\angle AOB \cong \angle DOE$

   (b) $\angle DOF \cong \angle BOF$

   (c) $\angle DOC \cong \angle FOG$

9. (a) Is the interior of a triangle a convex set?

   (b) Is the exterior of a triangle a convex set?

   (c) Given ΔRST and a point P. If P and R are on the same side of $\overleftrightarrow{ST}$ and if P and S are on the same side of $\overleftrightarrow{RT}$,

   (1) Is P in the interior of $\angle RTS$?

   (2) Is P in the interior of ΔRST?

10. Given a ray-coordinate system in a plane with coordinates assigned as in the figure to the right. Find the measure of each of the following angles:

(a) $\angle CAE$

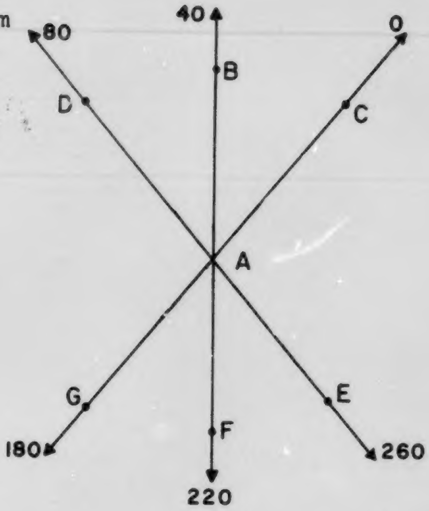
(b) $\angle CAD$

(c) $\angle BAE$

(d) $\angle DAB$

(e) $\angle DAG$

(f) $\angle CAF$


40
80
0
B
D
C
A
G
E
F
180
260
220

Chapter 5

CONGRUENCE

The treatment of congruence in this chapter will seem unfamiliar to many teachers, but the two Talks, Equality, Congruence, and Equivalence, and The Concept of Congruence, should be helpful to them. The difference in treatment lies chiefly in the fact that congruence is regarded here as a special kind of one-to-one correspondence, and in the distinction which is maintained between the properties of equality and the properties of congruence as logical bases for steps in a two-column proof.

Our definition of congruent triangles is essentially the conventional one: One triangle is a "copy" of the other in the sense that its parts are "copies" of the corresponding parts of the other. But observe the precision with which it is formulated. The correspondence doesn't depend on an individual's interpretation of the vague term "corresponding" but is based objectively on a pairing of the vertices

$$A \longleftrightarrow A', \quad B \longleftrightarrow B', \quad C \longleftrightarrow C',$$

which induces a pairing of sides and of angles

$$\overline{AB} \longleftrightarrow \overline{A'B'}, \quad \overline{BC} \longleftrightarrow \overline{B'C'}, \quad \overline{CA} \longleftrightarrow \overline{C'A'},$$

$$\angle A \longleftrightarrow \angle A', \quad \angle B \longleftrightarrow \angle B', \quad \angle C \longleftrightarrow \angle C' .$$

Spelling out the notion of "corresponding" in this way points up the importance of a one-to-one correspondence in the notion of a congruence more effectively than in the conventional treatment. Our treatment brings to the fore the idea of a one-to-one correspondence between the vertices of ΔABC and $\Delta A'B'C'$ in which corresponding sides and corresponding angles are congruent. The triangles are congruent if and only if such a correspondence between their vertices exists.

The properties of congruence include the three properties which belong to equivalence relations generally. Congruence is an equivalence relation. Other equivalence relations in this text are equality, similarity, proportionality, parallelism of lines, parallelism of planes, vectors, and possibly others. The three properties common to all of these equivalence relations are reflexivity, symmetry, and transitivity. Appreciation in depth of the significance of these properties is not expected of students at first meeting. They will use the properties, or some of them, as reasons for steps of two-column proofs, and will thereby gain understanding of them throughout the year.

We have included problems to familiarize the students with the new terminology; the rest of the problems in the chapter are familiar in type. In this book, as in most books, the students are expected to develop a working knowledge of proof by working with congruence of triangles.

Students should show progress, while studying this chapter, in their ability to recognize different proofs of a theorem. The tendency for them to think that a mathematical problem has only one method of solution should be replaced gradually by the practice of examining each proof as an example of correct logical reasoning.

The extent to which a proof is detailed is mainly a matter between teacher and student. We believe it desirable to develop flexibility of methods dependent upon the problem at hand and the mathematical maturity of the students involved. As the student progresses he should be encouraged to omit minor steps where understanding is not impaired and convenience results. For example, if the hypothesis of a theorem says that M is the midpoint of $\overline{AB}$, the teacher may require, in the first proofs the student does, that $AM = MB$ be justified in two steps:

1. M is the midpoint of $\overline{AB}$.	1. Hypothesis.
2. $AM = MB$.	2. Definition of midpoint.

As he learns, the student should be permitted to telescope this into one step by saying $AM = MB$, by definition of midpoint (or even, by hypothesis). The important thing is to advance the student's growth in the direction of appreciating and understanding proof. There are many exercises and examples in this chapter to help the student in this growth. We leave it to the teacher's discretion as to how many of these should be assigned. We point out that Chapters 6 and 7 also provide examples and problems requiring synthetic proofs that will contribute to the students' growth.

In our discussion of physical geometry, it seems to us that the experimental basis for S.A.S., A.S.A., S.S.S. congruence ideas are on a par with each other. For this reason we feel that it is better pedagogy at this stage to postulate all three. Of course, this is not necessary mathematically, since, for instance, A.S.A. and S.S.S. can be deduced as a consequence of the S.A.S. Congruence Postulate. The details of these deductions are included in an Appendix to Part I of the text.

This chapter is characterized by a thorough-going approach to proof. Since students learn by doing, we have provided a large number of problems. On the other hand, doing something repeatedly that has been mastered leads to boredom. The teacher should exercise his judgment as to what the class needs. Possible uses of the extra problems are to include them in future assignments or to use them as additional drill material for those students who need it.

228 The fact that many teachers are unfamiliar with this way of writing a correspondence may make it difficult for them. But the student at this point does not have a past experience to influence his likes and dislikes. Actually, it should be easier to teach them what is written on the line than to teach them to read between the lines.

A mechanical approach to one-to-one correspondence might give some assistance in writing correspondences.

If you know the correspondences between the three angles, you should have no trouble. For example, if $\angle A \longleftrightarrow \angle F$, $\angle B \longleftrightarrow \angle D$, $\angle C \longleftrightarrow \angle E$, the triangle designated by any combination of the letters on the left has a correspondence with the combination taken in the same order from the right. For example: $\triangle ABC \longleftrightarrow \triangle FDE$ or $\triangle BAC \longleftrightarrow \triangle DFE$.

Given the correspondences between the sides, $\overline{AB} \longleftrightarrow \overline{DF}$, $\overline{AC} \longleftrightarrow \overline{FE}$, $\overline{BC} \longleftrightarrow \overline{DE}$, we can determine the corresponding angles by considering the side correspondences two at a time. Take the first two; on the left side A is common, and on the right, F. So $A \longleftrightarrow F$. With the last two, $C \longleftrightarrow E$. The only possible third pair is $B \longleftrightarrow D$. Now state the triangle correspondence as above.

Suppose that we have a congruence where $\overline{RS} \cong \overline{LM}$, $\overline{ST} \cong \overline{MN}$ and $\angle RST \cong \angle LMN$. From the angle congruence we know that $S \longleftrightarrow M$, since S and M are vertices of the angles. Since $S \longleftrightarrow M$, then from $\overline{RS} \cong \overline{LM}$ we see that $R \longleftrightarrow L$ and, from $\overline{ST} \cong \overline{MN}$, $T \longleftrightarrow N$ to give $\triangle SRT \cong \triangle MLN$.

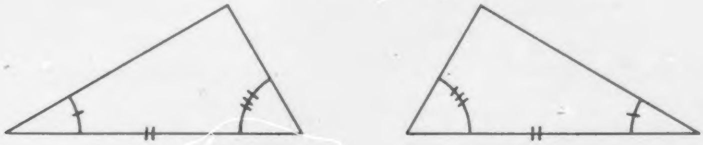
Notice that a statement that "two triangles are congruent" does not tell us which are the corresponding congruent parts, a piece of information that may be essential. All it says is that there is at least one correspondence in which corresponding parts are congruent. On the other hand our statement, $\triangle ABC \cong \triangle DEF$, not only says that the triangles are congruent, but it reveals a particular correspondence in which corresponding parts are congruent.

Contrast this with some conventional texts. In them, the statement, $\triangle ABC \cong \triangle DEF$ is written without regard to the order in which the letters are written. Clearly, our notation reveals more, and as you continue with the development, you will find that it makes many discussions, such as that of the Isosceles Triangle Theorem, elegantly simple.

230 To clarify the distinction between " $=$ " and " $\cong$ " let us skip ahead and examine the S.A.S. Postulate: "Given a one-to-one correspondence between the vertices of two triangles (not necessarily distinct). If two sides and the included angle of the first are *congruent* to the corresponding parts of the second triangle, then the correspondence is a congruence." Let us consider the word "congruent" that is underlined above. This *cannot* be replaced by "equals," since "equals" means "is the same as," and we would not be able to talk about two *different* triangles being congruent. Using "equals" we would be able to talk only about the same triangle which might become rather uninteresting. In the statement of the above postulate it is possible to replace the phrase, "are congruent to" by the phrase, "have the same measure as."

The question may very well arise as to why we have two different ways of writing exactly the same thing. If $\overline{AB} \cong \overline{CD}$ means that $AB = CD$, why bother to introduce the notation $\overline{AB} \cong \overline{CD}$? This would be a valid objection if we were talking about congruence of segments only. But we will be talking about congruence of segments, angles and triangles; and while the technical definitions of congruence are different for these three cases, the basic intuitive idea is the same. Notice that in the definitions of congruent angles and segments the idea of a one-to-one correspondence is not brought to the fore as it is in the development of the basic idea of a congruence between two triangles. The idea does appear, however, in the general definition of congruence given in the Appendix on Rigid Motion.

231 Students should be encouraged to mark corresponding parts of congruent triangles in some manner. Either the method indicated in these problems may be used, or the method indicated below might be used.



233 As already mentioned, equality and congruence share the properties of equivalence relations. However, equality has certain properties, notably the substitution, addition, and multiplication properties, which are not properties of congruence. The equality relation is fundamentally the relation of logical identity. We apply it in two situations: to identify two or more sets of points as one and the same set, and to identify two or more expressions as symbols for the same number. Thus we write $AB = AC$, implying that B and C are the same set of points. Also, we write $9 - x = 3$, implying that $x = 6$ because $9 - x$ and 3 are the same number. Similarly, we write $DE = FG$ meaning that the number which is the distance between D and E is the same as the number which is the distance between F and G. Since segments of equal measure are congruent, by definition, it follows in this case that $\overline{DE} \cong \overline{FG}$, but not that $\overline{DE} = \overline{FG}$ unless $\overline{DE}$ and $\overline{FG}$ name the same set of points.

(In Chapter 11, but not before, the distinction here indicated is relaxed to the extent that the word "equal" is permitted to signify "equal in length" and "equal in area.")

It may be useful to look ahead at certain situations where it would be appropriate to use a property of congruence or of equality to justify a step in a two-column proof.

One such situation occurs frequently. Suppose we wish to utilize the fact that two triangles have a common side. We write $\overline{AB} = \overline{AB}$ and cite as reason "reflexive property of equality," or $\overline{AB} \cong \overline{AB}$, "reflexive property of segment congruence," or $AB = AB$, "reflexive property of equality." Note that in the first case we assert the identity of sets of points, and in the third case, the identity of numbers.

Teachers will recall many situations which demand as reasons such conventional-text expressions as, "If equals are added to equals, the sums are equal." In this text, the addition and multiplication properties of equality fill such needs.

The purposes served by the conventional-text postulate, "If two quantities are equal to the same quantity, they are equal to each other" are served by the transitive property of equality (when dealing with measures) or of congruence (when dealing with sets of points).

Little use is made of the symmetric property of either equality or congruence in this text.

244 The most effective way to learn about two-column proofs is to do some. The full meaning of the text explanation in this section will become apparent as the students gain experience. Teachers may wish to go directly to the problems and use the explanation as reference material.

In the SMSG publication, "Studies in Mathematics, Vol. II" there is a very informative and easy-to-read chapter on logic. Teachers are urged to read it.

277 At this stage most students should have acquired some proficiency in writing two-column proofs and be ready to try their hands at writing the more readable essay or paragraph type of proofs.

As a general rule, students should be allowed to develop their own styles of presentation, restricted only by the need to be logical and convincing. Some students will prefer to stick with the two-column proof; others will welcome the relaxation in formality afforded by the essay proof. It may be that some students will develop a preference for a sort of composite form which exhibits only the big steps in the left column with an essay justification on the right for each big step.

A good test for the logical structure of a proof in paragraph form is to translate it into two-column form, and such translation is good drill occasionally.

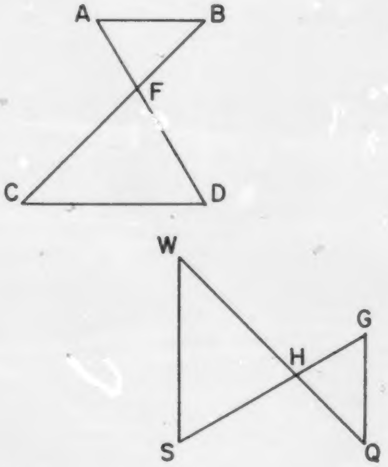
Illustrative Test Items for Chapter 5

The illustrative test items for this chapter include sufficient problems for several tests or quizzes. Teachers who wish to give tests or quizzes before the isosceles triangle theorems are introduced should select problems from the following list: Section A, Section B (1-6), Section C (2 and 6), Section D (1), Section E (2, 3, 5, 6, 8, 9, 10, 12, 13, 14).

A. 1. Below are listed the six pairs of corresponding parts of two congruent triangles. Name the congruent triangles.

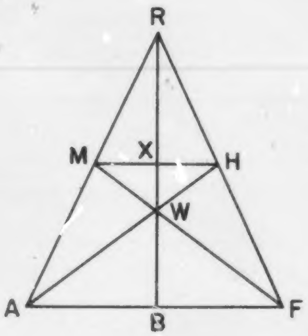
$\overline{AB} \cong \overline{MK}$. $\angle A \cong \angle M$.
$\overline{BW} \cong \overline{KF}$. $\angle B \cong \angle K$.
$\overline{AW} \cong \overline{MF}$. $\angle W \cong \angle F$.

2. The two plane figures at the right are congruent. Complete each correspondence in such a way that a congruence results.



(a) ABCD$\longleftrightarrow$ _______.
(b) BFA$\longleftrightarrow$ _______.
(c) FCD$\longleftrightarrow$ _______.
(d) ABFCD$\longleftrightarrow$ _____.

3. The plane figure at the right has several pairs of overlapping triangles. In each case find a triangle (different from the given one) which will form a correspondence that appears to be a congruence.



(a) ΔAFM$\longleftrightarrow$ ________.
(b) ΔAHR$\longleftrightarrow$ ________.
(c) ΔHFM$\longleftrightarrow$ ________.

B. 1. Given the figure shown below with $\triangle ABC \cong \triangle DEF$, and M between B and C. Write true if the statement is true. If the statement is not true, correct it so that it will be true.

(a) $\overline{AB} \cong \overline{DE}$.

(b) $\angle A = \angle D$.

(c) $BC = EF$.

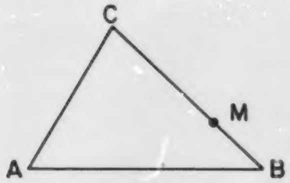
(d) $m\angle B = m\angle E$.

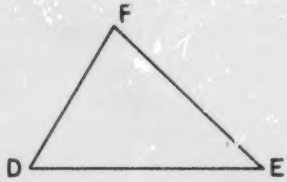
(e) $\angle ABC = \angle ABM$.

(f) $\angle ABC \cong \angle ABM$.

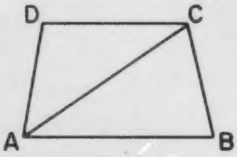
(g) $\angle C \cong \angle F$.

(h) $\angle ACB \cong \angle DEF$.





2. Given quadrilateral ABCD with diagonal $\overline{AC}$. Fill in the blanks to complete the listed combination.



(a) side, angle, side of $\triangle ACD$: $\overline{AC}$, ____, $\overline{AD}$.

(b) angle, side, angle of $\triangle ABC$: ____, $\overline{AB}$, ____.

3. Complete the following definitions:

(a) Two angles are congruent if ________.

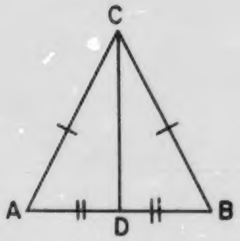
(b) Two segments are congruent if _______.

(c) An _____ triangle is one having two congruent sides.

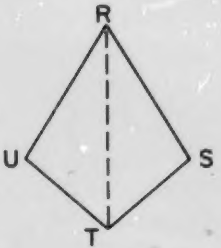
(d) $\angle XYZ$ is bisected by a ray $\overrightarrow{YS}$ if S is in ________ and if __________.

(e) A segment whose endpoints are a midpoint of one side of a triangle and the opposite vertex is a _________________ of the triangle.

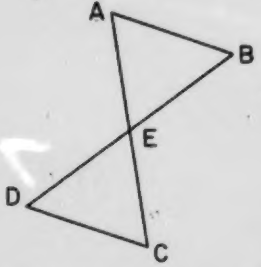
4. In ΔABC as marked in the figure, $\overline{CD}$ is __________ to the base of the triangle and $\angle ACB$ is the ________ of the triangle.



5. (a) "If $RS = RU$ and $ST = UT$, then $\angle RUT \cong \angle RST$." Will the same proof hold regardless of whether or not R is coplanar with S, T, and U ?

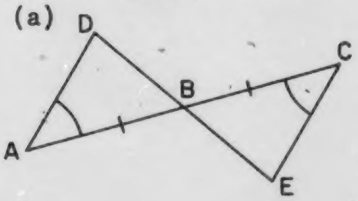


(b) "If E is the midpoint of $\overline{AC}$ and $BE = DE$, can we prove that $\Delta ABE \cong \Delta CDE$ if D and C are not in the same plane as points A, B, and E. Explain why.



6. Consider the pairs of triangles pictured below. Like markings indicate congruent parts; all points are coplanar; and points which appear to be collinear are collinear. From this information and the information which can be deduced at the present time, some of the pairs of triangles can be proved congruent and others cannot be proved congruent. In each of the problems give the abbreviated congruency statement which justifies the congruence of the triangles or write "I. E." if there is insufficient evidence to prove the triangle congruent.

(a)



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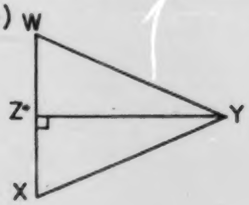

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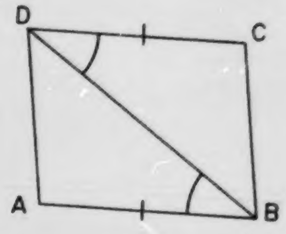

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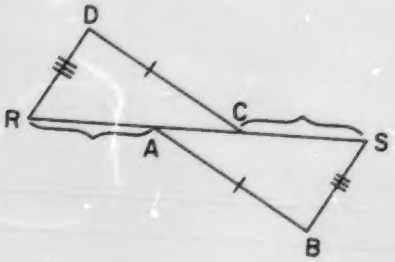
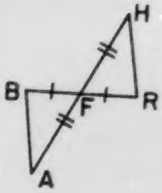


(e)



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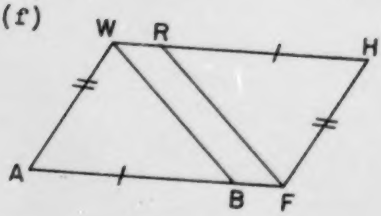

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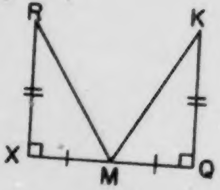
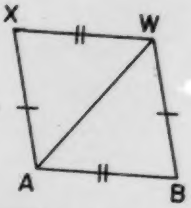
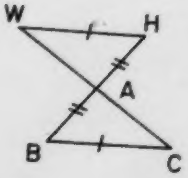


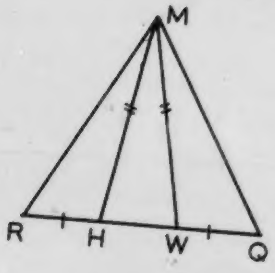
(h)



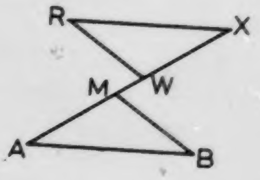
(i)



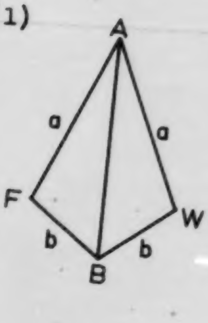
(j) Consider Δ RWM and Δ QHM



(k) AW = XM, AB = XR, ∠A ≅ ∠X

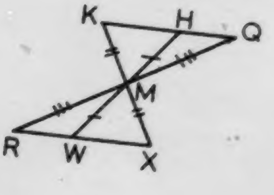


(l)



Consider: (m) Δ RMW and Δ QMH

(n) Δ WMX and Δ HMK



7. In each of the following, if enough information is given to establish a congruence between the two triangles, state the appropriate reason by writing S.A.S., S.S.S., or A.S.A. If insufficient information is given, name one other pair of corresponding parts which, if congruent, would enable you to prove the triangles congruent. All points are coplanar and points which appear to be collinear are collinear. In Figure 1:

(a) $\angle ADB \cong \angle CDB$, $\overline{AD} \cong \overline{CD}$.

(b) $\overline{AB} \cong \overline{CB}$.

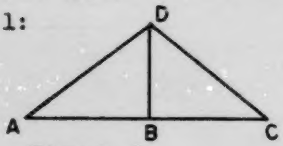


Figure 1

In figure 2:

(c) $UT = ST$, $VT = RT$.

(d) $UV = RS$, $UT = ST$.

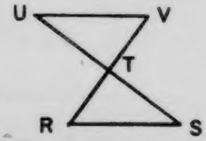


Figure 2

In figure 3:

(e) $\angle JFG \cong \angle HFG$, $\angle HGF \cong \angle JGF$.

(f) $FJ = FH$, $JG = HG$.

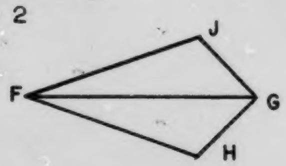


Figure 3

8. The information given in the following statements refers in each case to $\triangle ABC$ with point D between A and B. If the given information is sufficient to prove $\triangle ADC \cong \triangle BDC$, write the abbreviation of the congruence statement which would be used as the reason in the last step of the proof. Otherwise write "I. E." for insufficient evidence. Make use of any congruences which follow from the given information using any theorems, postulates, or definitions which we have had.

(a) $AC = BC$, $AD = DB$.

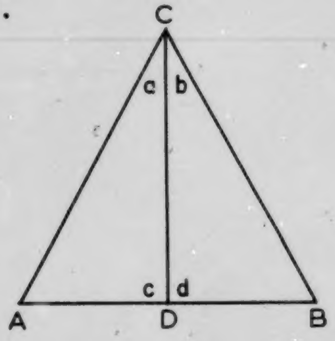
(b) $\overline{AC} \cong \overline{BC}$, $\angle a \cong \angle b$.

(c) $\angle a \cong \angle b$, $\angle c \cong \angle d$.

(d) $AC = BC$, $\angle A \cong \angle B$.

(e) $\overline{AD} \cong \overline{DB}$, $m\angle c = m\angle d$.

(f) $\overrightarrow{CD}$ bisects $\angle C$.



(g) $\overline{CD} \perp \overline{AB}$.

(h) $\overline{CD}$ is a median to $\overline{AB}$.

(i) $AC = BC$, $\overrightarrow{CD}$ bisects $\angle C$.

(j) $\overline{CD} \perp \overline{AB}$, $\overrightarrow{CD}$ is the bisector of $\angle C$.

(k) $\angle ACD \cong \angle BCD$, $\angle CAD \cong \angle CBD$.

(l) $\overrightarrow{CD}$ bisects $\overline{AB}$, $\overline{AC} \cong \overline{CB}$.

(m) $\angle a \cong \angle b$, $\angle c \cong \angle d$, $\angle A \cong \angle B$.

9. Indicate whether each of the following is true or false:

(a) If $\triangle ABC \cong \triangle CAB$, then $\angle A \cong \angle B$.

(b) All equilateral triangles are congruent to each other.

(c) Given a correspondence between the vertices of two triangles such that two sides and an angle of the first triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

(d) If $\angle ABC \cong \angle XYZ$, then the points A, B, and C coincide respectively with points X, Y, and Z.

(e) An equilateral triangle is isosceles.

10. State whether or not each pair of triangles described below can be proved congruent using postulates and theorems we have had.

(a) Two isosceles triangles with congruent bases.

(b) Two equilateral triangles with congruent bases.

(c) Two isosceles triangles with congruent bases and a base angle of one congruent to a base angle of the other.

(d) Two isosceles triangles with congruent vertex angles.

C. 1. In each of the following statements write the word or words which completes it correctly.

(a) Angles are congruent if they have the same ______.

(b) If two sides of a triangle are ______, then the angles ______ those sides are ____________.

(c) Segments are ______ if they have the same length.

(d) If the vertices of two triangles are in correspondence so that every pair of corresponding angles are ________ and every pair of corresponding ______ are congruent then the correspondence is a ________ between the triangles.

(e) The abbreviations for the triangle congruence postulates are ________, _________, and __________.

(f) If a triangle is congruent to itself using a correspondence in a congruence in which at least one vertex does not correspond to itself, then the triangle is an _________ triangle; it may be an _________ triangle.

(g) Through a given external point, there exixt(s) at most ______ line(s) perpendicular to a given line.

(h) In an isosceles triangle the ______ angle is the angle included between its congruent sides.

(i) A segment whose endpoints are the vertex of the triangle and the midpoint of the opposite side is called a __________.

(j) The measure of an exterior angle of a triangle is always _________ the measure of either of its nonadjacent interior angles.

2. Match each word in Column B to a description in Column A.

Column A

(a) A statement assumed to be true.

(b) A chain of statements which show how the conclusion follows logically from the hypothesis.

(c) An agreement that the names of a thing are interchangeable.

(d) A statement that follows logically from another statement or combination of statements.

(e) A property of equality which states that for all c, $c = c$.

(f) A statement which supports a step of a proof.

(g) A statement which requires proof.

(h) A part (or all) of the "if clause" in a conditional statement.

Column B

(1) deduction

(2) substitution property

(3) theorem

(4) postulate

(5) reflexive

(6) proof

(7) hypothesis

(8) reason

3. Rewrite each of the following definitions in complete form.

(a) Angles with the same measure are said to be congruent.

(b) The lines determined by two rays which form a right angle are called perpendicular lines.

4. Write the converse of each of the following statements and indicate whether that converse is true.

(a) If two angles form a linear pair, they are supplementary.

(b) If two angles are congruent, they are right angles.

(c) If $x + 4 = 9$, then $x = 5$.

(d) Vertical angles are congruent.

(e) If $y < 0$, then y is a negative integer.

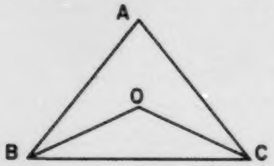
(f) Segments which have the same length are congruent.

5. Indicate whether each of the following statements is true or false.

   (a) Any definition can be written in two parts.
   (b) If a converse of a statement is false, the statement must be false.
   (c) A proof need not be written in two-column form.
   (d) A statement in the hypothesis of a theorem may be used in a proof of the theorem.
   (e) The addition property of equality does not pertain to statements like: If $a = x$ and $b = y$, then $a - b = x - y$.

6. Select the reason from Column A which should be used to justify the conclusion in each of the following problems.

   In the figure below we are given ΔABC with point O such that $\overrightarrow{BO}$ is between $\overrightarrow{BC}$ and $\overrightarrow{BA}$, and $\overrightarrow{CO}$ is between $\overrightarrow{CB}$ and $\overrightarrow{CA}$.



Column A

(1) Substitution property of equality.
(2) Transitive property of equality.
(3) Symmetric property of equality.
(4) Reflexive property of equality.
(5) Betweenness addition theorem.
(6) Multiplication property of equality.

   (a) If $\overline{AB} \cong \overline{AC}$, then $\overline{AC} \cong \overline{AB}$.
   (b) If $\overline{BC} \cong \overline{AB}$ and $\overline{AB} \cong \overline{AC}$, then $\overline{BC} \cong \overline{AC}$.
   (c) If $\angle CBA \cong \angle BCA$ and $\angle CBO \cong \angle BCO$, then $\angle ABO \cong \angle ACO$.
   (d) If $\overrightarrow{BO}$ bisects $\angle CBA$ and $\overrightarrow{CO}$ bisects $\angle BCA$ and $m\angle CBO = m\angle BCO$, then $m\angle CBA = m\angle BCA$.
   (e) If $\overrightarrow{BO}$ bisects $\angle CBA$, $\overrightarrow{CO}$ bisects $\angle BCA$, and $m\angle CBA = m\angle BCA$, then $m\angle CBO = m\angle BCO$.

(Problem 6 continued on following page)

6. (continued)

(f) $\overline{BC} \cong \overline{CB}$.

(g) If $\angle CBO \cong \angle BCO$, $\angle OBA \cong \angle OCA$, then $\angle CBA \cong \angle ACB$.

(h) If $m\angle CBO = m\angle OBA$, $m\angle BCO = m\angle OCA$, and $m\angle CBO + m\angle BCO + m\angle BOC = 180$, then $m\angle OBA + m\angle OCA + m\angle BOC = 180$.

Column A

(1) Substitution property of equality.

(2) Transitive property of equality.

(3) Symmetric property of equality.

(4) Reflexive property of equality.

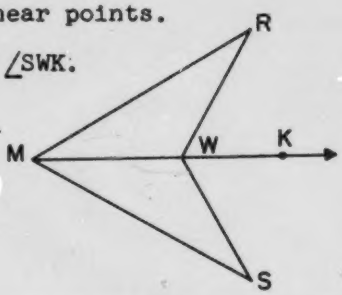
(5) Betweenness addition theorem.

(6) Multiplication property of equality.

D. 1. Hypothesis: M, W, K are collinear points.

$\angle RMW \cong \angle SMW$; $\angle RWK \cong \angle SWK$.

Prove: $\angle R \cong \angle S$.



Proof: (Supply the reasons.)

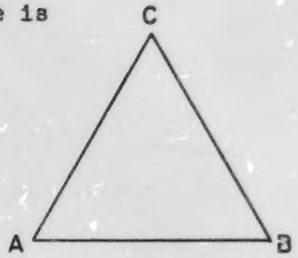
1. $\angle MWR$ is supplementary to $\angle RWK$, $\angle MWS$ is supplementary to $\angle SWK$.	1.
2. $\angle RMW \cong \angle SMW$. $\angle RWK \cong \angle SWK$.	2.
3. $\angle MWR \cong \angle MWS$.	3.
4. $\overline{MW} \cong \overline{MW}$.	4.
5. $\Delta MWR \cong \Delta MWS$.	5.
6. $\angle R \cong \angle S$.	6.

2. Study the formal proof of the following corollary, then write the proof in paragraph form.

Every equilateral triangle is also equiangular.

Hypothesis: Given ΔABC,
$BC = AC = AB$.

Prove: $\angle A \cong \angle B \cong \angle C$.

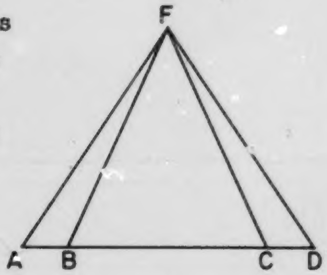


1. $BC = AC$.	1. Hypothesis.
2. $\angle A \cong \angle B$.	2. Base angles of an isosceles triangle are congruent.
3. $AC = AB$.	3. Hypothesis.
4. $\angle B \cong \angle C$.	4. Base angles of an isosceles triangle are congruent.
5. $\angle A \cong \angle B \cong \angle C$.	5. Transitive property of congruence.

E. Write a proof for each of the following problems.

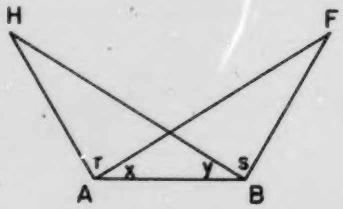
1. Hypothesis: In ΔADF, the points A, B, C, and D are collinear in that order. $FA = FD$ and $AB = DC$.

Prove: (a) $\Delta AFB \cong \Delta DFC$.
(b) $\angle FBC \cong \angle FCB$.



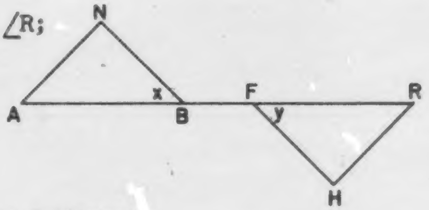
2. Hypothesis: In the figure to the right, A, B, F, and H are coplanar points. $AH = BF$; $\angle r \cong \angle s$; $\angle x \cong \angle y$.

Prove: $HB = FA$.



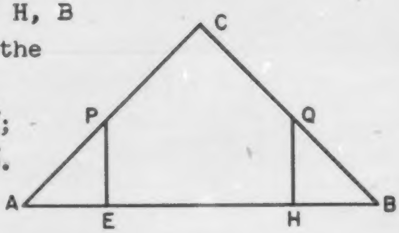
3. Hypothesis: A, B, F, R are collinear in the order given. $AF \cong RB$; $\angle A \cong \angle R$; $\angle x \cong \angle y$.

Prove: $\overline{AN} \cong \overline{RH}$.



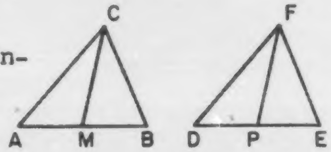
4. Hypothesis: In ΔABC, P is between A and C; Q is between B and C; A, E, H, B are collinear in the order given. $\overline{AC} \cong \overline{BC}$; $\overline{AE} \cong \overline{BH}$; $\overline{PE} \perp \overline{AB}$; $\overline{QH} \perp \overline{AB}$.

Prove: $PE = QH$.



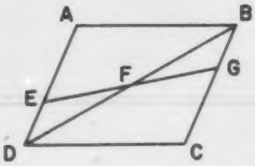
5. Hypothesis: In ΔABC and ΔDEF, $AC = DF$; $AB = DE$; $\overline{CM}$ and $\overline{FP}$ are congruent medians.

Prove: $\Delta ABC \cong \Delta DEF$.



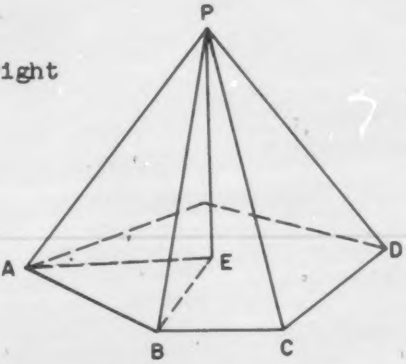
6. Hypothesis: In quadrilateral ABCD, E is between A and D; G is between B and C. $AB = CD$; $AD = CB$; and F bisects $\overline{BD}$.

Prove: $EF = GF$.



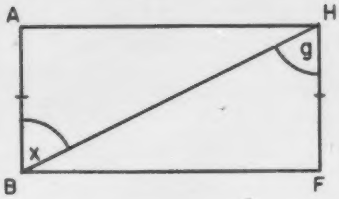
7. In the pyramid at the right $\overline{PE} \perp \overline{EA}$; $\overline{PE} \perp \overline{EB}$; $EA = EB$.

Prove: $\angle PAB \cong \angle PBA$.



8. In the figure to the right, the points A, B, H, and F are coplanar. AB = FH and m∠x = m∠g.

Prove: m∠A = m∠F;
m∠ABF = m∠FHA.



9. In Problem 8 assume that A, B, H, and F are not coplanar points.

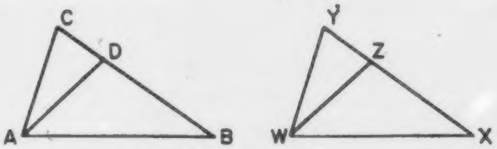
(a) Is m∠A = m∠F ?
(b) Is m∠ABF = m∠FHA ?

10. Prove that in quadrilateral ABCD if AB = BC = CD, if ∠B and ∠C are right angles, and if E is the midpoint of $\overline{BC}$, then $\overline{DE} \cong \overline{AE}$.

11. Prove the theorem that the median from the vertex of an isosceles triangle determines the bisector of the vertex angle of the triangle.

12. Hypothesis: △ABC ≅ △WXY .
$\overrightarrow{AD}$ and $\overrightarrow{WZ}$ are angle bisectors.

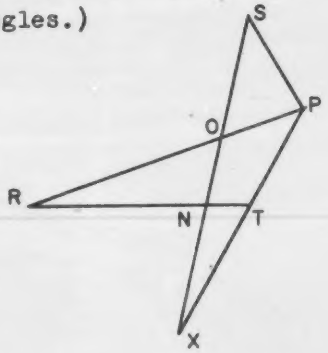
Prove: $\overline{AD} \cong \overline{WZ}$.



13. Prove: The ray containing the diagonal of a square bisects its angles. (Note: A square is defined as a quadrilateral having four congruent sides and four congruent angles.)

14. In the figure,
Hypothesis: ∠RTP ≅ ∠XPS;
$\overline{PT} \cong \overline{SP}$; and
∠PSO ≅ ∠TPO.

Prove: $\overline{RT} = \overline{XP}$



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Chapter 6

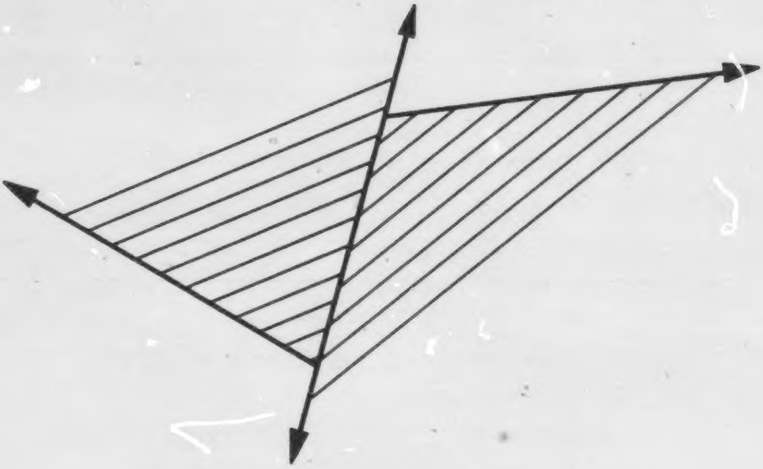
PARALLELISM

In this chapter, we depart from the usual definition of parallel lines which is somewhat limited in scope and define parallel lines in such a way that permits a line to be parallel to itself. For example if p and q are two coplanar lines which do not intersect, then p and q are both distinct and parallel. If p and q intersect in at least two points, then according to our definition, the lines are parallel, but they are not distinct. This treatment of parallel lines is consistent with the trend in modern mathematics and will be especially convenient for the student in his work with coordinates and vectors.

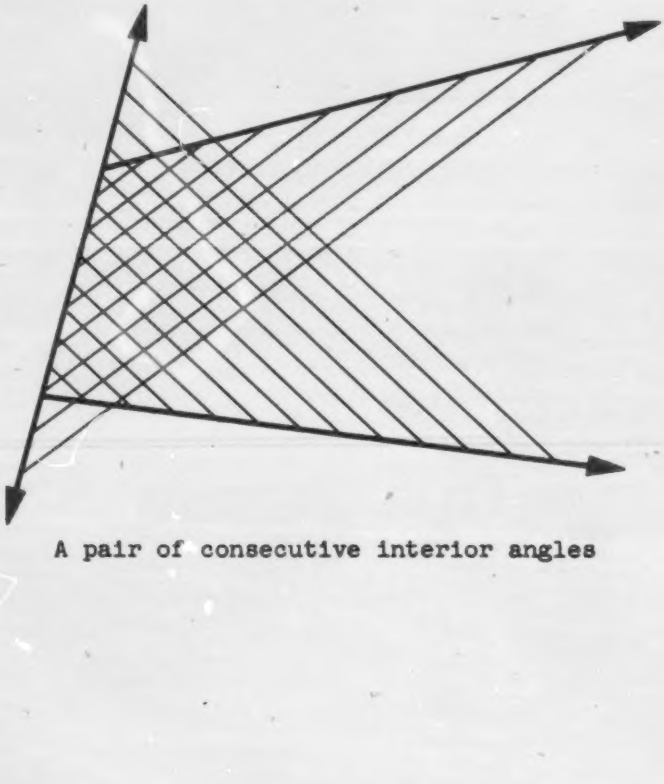
To help the students gain a feeling for three dimensions, call attention to the existence of skew lines by asking them to demonstrate with two pencils the case where two distinct lines are not coplanar (in which case they of course do not intersect).

Remind students that distinct parallel lines do not meet. You will sometimes hear the expression: "Parallel lines meet at infinity." This does not mean that the lines do meet.

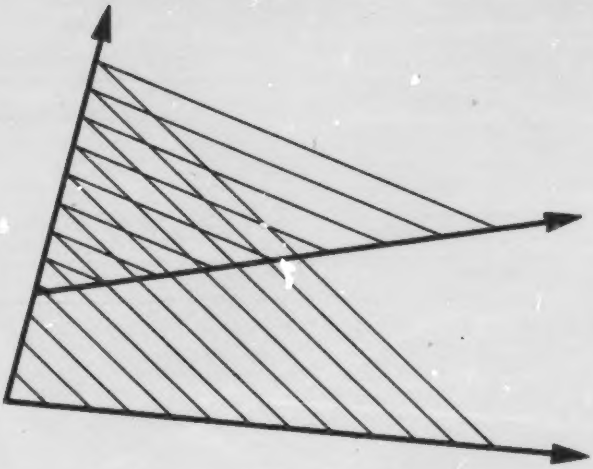
The definitions of transversal, alternate interior angles, consecutive interior angles, and corresponding angles differ from the usual definitions of these terms. The use of union and intersection help us to simplify these definitions and to be precise as to the meaning of the terms. In discussing these definitions with the class, teachers should find it helpful to ask students to use diagonal lines as illustrated in the diagrams below to shade the interiors of the various angle pairs. This should make the definitions more meaningful to the student.



A pair of alternate interior angles



A pair of consecutive interior angles



A pair of corresponding angles

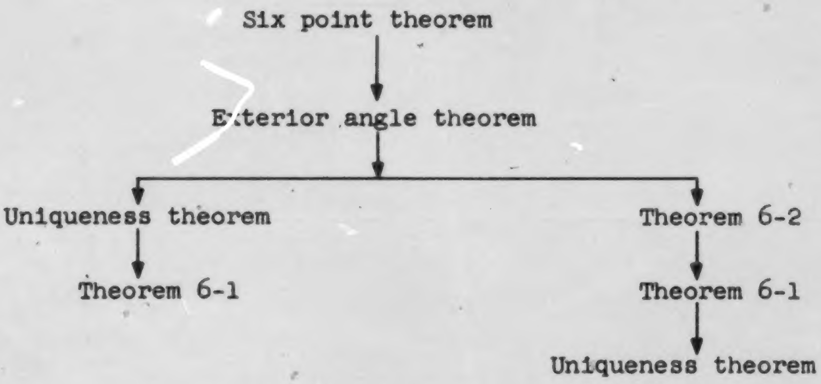
The statements which follow the definitions are a direct outgrowth of the definitions and will offer the student another method of identifying these angles.

Note that the definition of a transversal is for two distinct coplanar lines. No mention is made of a transversal of noncoplanar lines since it is not needed in the development of our geometry.

The theorems in Section 6-3 are similar to those in most conventional texts. It should be noted that these theorems depend upon the exterior angle theorem which has been strengthened by the proof of the six point theorem in Chapter 4, (Theorem 4-7). It is also interesting to note that Theorem 6-1 can be proved as a corollary to Theorem 6-2. When treated in this manner, the proof implies the uniqueness of the perpendicular to a given line from a point not on the line. Our reason for not using this sequence is that we believe students will be better able to understand the uniqueness of the perpendicular if it is discussed in a separate setting immediately following the proof of the exterior angle theorem. In addition to this, the use of the uniqueness theorem makes the proof of Theorem 6-1 a very easy indirect proof and thus serves as an ideal method of using parallel lines in reviewing indirect proof at the beginning of the chapter.

Teachers will probably find it desirable to ask students to reread the proofs of the six point theorem, the exterior angle theorem, and the uniqueness theorem before proceeding with the work in Section 6-3.

The following diagram should help teachers to visualize the relations that exist among these theorems.



The Property of the Contrapositive is discussed in the section on indirect proof. This property permits the student to state immediately that "If p , then q" is a true statement, then "If not q , then not p" is also a true statement. This technique is illustrated in the proof of Theorem 6-2. The first four steps of this proof prove the contrapositive of the theorem. This is logically equivalent to the restatement of the theorem in Steps 5 and 6. It is important to note in Step 5 that we do not know by hypothesis which pair of alternate interior angles are congruent. However, we do know that if any one pair is congruent by the hypothesis, then the other pair is congruent by supplements of congruent angles. In writing the proof of this theorem, some teachers might prefer to omit Step 5 and write Property of the Contrapositive as the reason for $p \parallel q$.

Section 6-5 attempts to show the historical importance of the Parallel Postulate and to give the student a better understanding and greater appreciation of the postulational approach. It is important for students to understand that postulates are not self-evident truths; instead they are initial statements which we assume to be true and from which other statements can be deduced. Our Parallel Postulate and the parallel postulates of Non-Euclidean geometry should lead the student to understand that different geometries can be developed as a result of accepting one or the other of a pair of contradictory postulates; that although these geometries will differ from each other, they may remain consistent within themselves.

References are listed for students who are interested in Non-Euclidean geometries. However, time does not permit us to give more than a casual acquaintance of the subject in the text. Teachers should strengthen their backgrounds on this subject by reading Introduction to Non-Euclidean Geometry, one of the talks to teachers, in Part III of the Commentary for Teachers.

The theorems in Section 6-6 are based on the Parallel Postulate. These theorems, except for Corollary 6-5-2, are included in most standard textbooks. Since Corollary 6-5-2 is necessary for the development of coordinates in a plane, we include it at this time. It might conveniently be referred to as the graph paper corollary.

In Section 6-7 parallelism for segments is established, and a parallelogram is defined. Since many theorems relating to parallelograms can be conveniently proved by coordinate methods, we prove only two theorems for the parallelogram which are essential to the development of the coordinate plane. A more thorough treatment of parallelograms and trapezoids comes in Chapter 8 when students will have two methods of proof, rather than a single method of proof, at their disposal. The definition of distance between two parallel lines and the theorem that parallel lines are everywhere equidistant is a direct consequence of Theorem 6-6.

The notion of parallelism and anti-parallelism for rays is introduced in this section. The definition is

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consistent with our definition of parallel segments and parallel lines and further implies "same direction" for parallel rays and "opposite direction" for anti-parallel rays. These terms make possible a precise method of summarizing the results of Problems 9 through 12 in Problem Set 6-7 without the traditional reference to the right side and the left side of an angle. This concept is also useful in the study of vectors.

Sections 6-8 and 6-9 develop the theorem for the sum of the measures of the angles of a triangle and the usual corollaries and theorems relating to it. We have also included a theorem pertaining to the sum of the measures of the interior angles of a convex quadrilateral. This is needed in Chapter 8. The more general theorems concerning the sum of the measures of the interior and the exterior angles of any convex polygon is left to Chapter 11. In many texts, this work follows the theorem on the sum of the measures of the angles of a triangle. However, since our purpose is to give only the essential theorems relating to the development of the Pythagorean Theorem, we omit the work at this time and discuss it in relationship to polygonal-regions in Chapter 11.

In Section 6-10, it should be noted that $\{x,y\}$ is not an ordered pair of numbers. If a correspondence between $\{2,1\}$ and $\{0,10\}$ is given by $2 \longleftrightarrow 10$ and $1 \longleftrightarrow 0$, then the numbers 2,1 and 0,10 are unequal in the same order. This work is basic to an understanding of the theorems in Section 6-11 leading to the proof of the Triangle Inequality Theorem.

Problem Solutions for Chapter 6

In reading the solutions to the problems in this chapter, you should note that in some cases only an outline or sketch of a proof is given. In other cases, a complete proof of the problem is given either in paragraph form or two-column form.

Illustrative Test Items for Chapter 6

I. Write + if the statement is true and 0 if the statement is false.

1. The measure of an exterior angle of a triangle is greater than the measure of any interior angle of the triangle.
2. There is only one line perpendicular to a given line through a given point not on the line.
3. The angle opposite the longest side of a triangle has the greatest measure of any of the angles of the triangle.
4. In $\triangle ABC$, if $m\angle A > m\angle B$, then $AC > BC$.
5. If $\overline{AB} \perp \overline{BC}$, then $AB < AC$.
6. There is a triangle with sides having respective lengths 8 , 3 , and 12 .
7. If the measure of an angle of one triangle is greater than the measure of an angle of a second triangle, then the side opposite the angle in the first is longer than the side opposite the angle in the second.
8. Two lines in space are parallel if they are both perpendicular to the same line.
9. Through every point, there is a line parallel to a given line.
10. Given two lines and a transversal, if the angles in one pair of alternate interior angles are congruent, then the angles in the other pair are congruent also.
11. Given two coplanar lines and a transversal of these lines, the lines are perpendicular if the measure of one of the alternate interior angles is 90 .
12. If two coplanar lines and a transversal of these lines are given, then there are exactly two pairs of alternate interior angles.

13. If two lines intersect and a transversal of these lines is given, no alternate interior angles are congruent.

14. If two coplanar lines and a transversal of these lines are given such that a pair of consecutive interior angles are not supplementary, then the two lines intersect.

15. If two parallel lines and a transversal of these lines are given, then two consecutive interior angles are complementary.

16. If a, b, and c are three coplanar lines such that $a \parallel b$ and $b \parallel c$, then $a \parallel c$.

17. If a, b, and c are three lines such that $a \perp b$ and $b \perp c$, then $a \perp c$.

18. Since the sum of the measures of the angles of any triangle is three times 60, the sum of the measures of the angles of any quadrilateral is 4 times 60.

19. If two angles of one triangle are congruent respectively to two angles of another triangle, then the third angles are congruent.

20. The acute angles of a right triangle are complementary.

21. Given a one-to-one correspondence between the vertices of two triangles, if two angles and a side of one triangle are congruent respectively to the corresponding parts of the other triangle, then the correspondence is a congruence.

22. An exterior angle of a triangle is a supplement of one of the interior angles of the triangle.

23. If the union of a convex quadrilateral and one of its diagonals is the union of two congruent triangles (with a common side), then the quadrilateral is a parallelogram.

24. The union of a parallelogram and one of its diagonals is a set which is the union of two congruent triangles (with a common side).

25. Given a one-to-one correspondence between the vertices of two triangles, if two sides and an angle of one triangle are congruent respectively to the corresponding parts of the other triangle, then the correspondence is a congruence.

II. Make each of the following statements a true statement by writing either sometimes, always, or never:

1. In a given plane, if two lines are perpendicular to the same line, they are __________ perpendicular to each other.

2. If two coplanar lines are parallel to the same line, they are __________ parallel to each other.

3. If a line is parallel to one of two perpendicular lines, it is __________ parallel to the other.

4. If two lines are perpendicular to the same line, they are ___________ parallel to each other.

5. If two lines do not intersect, they are _____________ parallel to each other.

6. Given two coplanar lines and a transversal of these lines, a pair of corresponding angles are ___________ congruent.

7. If two lines are parallel, they are _______________ coplanar.

8. In a given plane, if a line intersects one of two parallel lines in a single point, it ______________ intersects the other.

9. Two lines and a transversal of those lines are ___________ coplanar.

10. If a line intersects one of two perpendicular lines, it __________ intersects the other.

11. Two antiparallel rays are __________ distinct.

12. If a pair of consecutive interior angles determined by two coplanar lines and a transversal are not supplementary, the lines are ____________ parallel.

III. Solve each of the following problems:

1. A pair of consecutive interior angles formed by two parallel lines and a transversal of these lines have measures equal to n and $(24 + n)$. Find the measures of the two angles.

2. Given: $\angle a$ and $\angle b$ are a pair of consecutive interior angles determined by two parallel lines and a transversal. If $m \angle a = \frac{3}{2} m \angle b$, find $m \angle a$ and $m \angle b$.

3. Two parallel lines and a transversal are given. The measure of one angle in a pair of consecutive interior angles exceeds the measure of the other by 20 . Find the measure of each of these angles.

4. If $m \angle a = 52$ and x is parallel to y , find $m \angle b$. (See Figure a.)

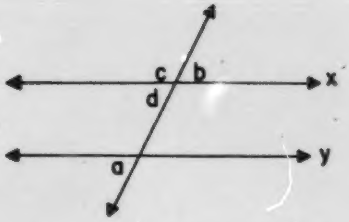


Figure a

5. If $m \angle a = 47$ and $m \angle c = 143$, is x parallel to y ? (See Figure a.)

6. In Figure b, the lines p and q are parallel. If $m \angle X = 30$ and $m \angle Y = 100$, find $m \angle Z$.

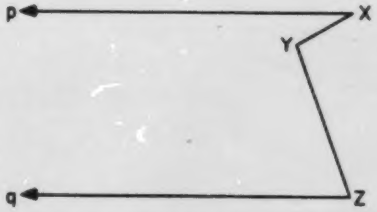


Figure b

7. In Figure c, the lines p and q are parallel. If m ∠X = 120 and m ∠Y = 100 , find m ∠Z .

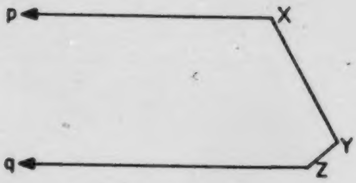


Figure c

8. In Figure d, points A, C, D are collinear in that order. $\overrightarrow{AE}$ is a midray of ∠CAB and $\overrightarrow{CE}$ is a midray of ∠DCB . If m ∠B = 100 and m ∠CAB = 50 , find m ∠E .

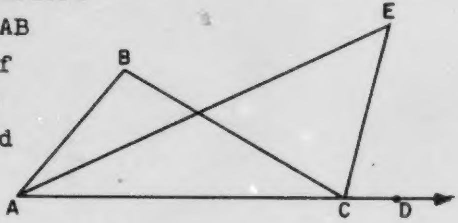


Figure d

9. In Figure e, ∠c is an exterior angle of △ABC .
m ∠a = 2x ;
m ∠b = x + 50 ;
m ∠c = 5x - 20 .

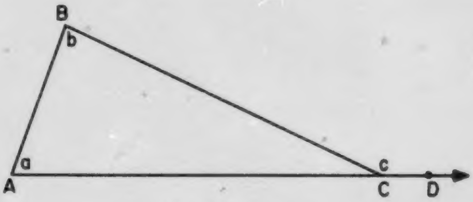


Figure e

(a) Find:
  (1) x ;
  (2) m ∠a ;
  (3) m ∠b ;
  (4) m ∠c ;
  (5) m ∠BCA .
(b) Indicate whether the following are true or false:
  (1) AB > AC ;
  (2) AC > BC ;
  (3) AC > AB .

10. In Figure f, the points R, A, C, and D are collinear in that order. m ∠d = m ∠e ; m ∠c = 78 ; m ∠d = 35 .

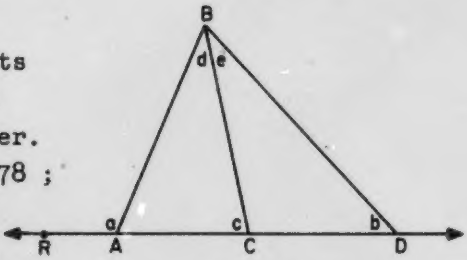


Figure f

(a) Find: (1) m ∠a
  (2) m ∠b
(b) In △ABD , the order relations among the sides are _____ > _____ > _____ .

11. In Figure g, ABEF is a convex quadrilateral. C is between A and F, D is between B and E, G is between C and D, and H is between F and E. In each of the following cases, tell what lines are parallel as a result of the given condition. If no lines can be proved parallel, write none.

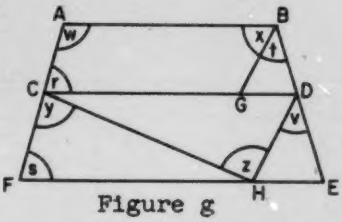


Figure g

(a) Hypothesis: m $\angle r$ = m $\angle s$.
Then ________________ .

(b) Hypothesis: m $\angle r$ = m $\angle t$.
Then ________________ .

(c) Hypothesis: m $\angle t$ = m $\angle v$.
Then ________________ .

(d) Hypothesis: $\angle y \cong \angle z$.
Then ________________ .

(e) Hypothesis: $\angle r \cong \angle x$.
Then ________________ .

(f) Hypothesis: $\angle w$ is a supplement of $\angle x$.
Then ________________ .

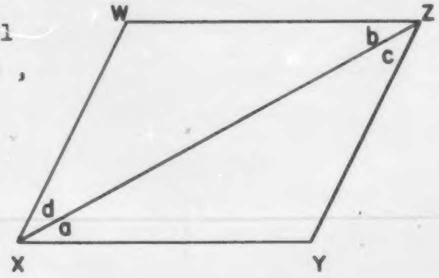
(g) Hypothesis: $\overline{AB} \parallel \overline{CD}$;
$\overline{FE} \parallel \overline{CD}$.
Then ________________ .

(h) Hypothesis: $\overline{AC} \perp \overline{CH}$;
$\overline{DH} \perp \overline{CH}$.
Then ________________ .

IV. In each of the following problems, choose the correct answer from the four listed in the problem. Each problem is accompanied by a diagram.

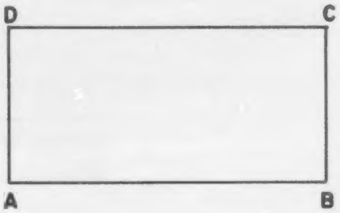
1. In convex quadrilateral XYZW , if $m\angle a = m\angle b$, then:

   (a) $m\angle c = m\angle d$.
   (b) $\overleftrightarrow{XY} \parallel \overleftrightarrow{WZ}$.
   (c) $\overleftrightarrow{WX} \parallel \overleftrightarrow{ZY}$.
   (d) $m\angle W = m\angle Y$.



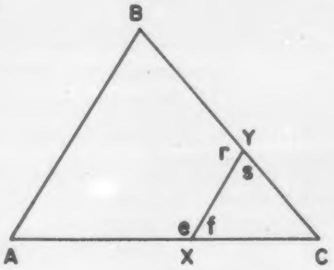
2. In the quadrilateral ABCD , if $\overline{CD} \perp \overline{AD}$ and if $\overline{AB} \perp \overline{AD}$, then

   (a) $\overline{DC} \perp \overline{BC}$.
   (b) $\overline{AB} \perp \overline{CB}$.
   (c) $\overline{AB} \parallel \overline{CD}$.
   (d) $\overline{AD} \parallel \overline{BC}$.



3. In triangle ABC , $\overleftrightarrow{XY} \parallel \overleftrightarrow{AB}$. Which one of the following conclusions is not deducible from the given information?

   (a) $\angle A$ is the supplement of $\angle e$.
   (b) $m\angle A = m\angle f$.
   (c) $m\angle A = m\angle B$.
   (d) $m\angle B = m\angle s$.



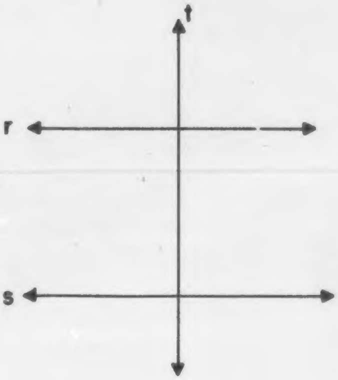
4. In a plane, if $r \perp t$ and $s \perp t$, which of the following statements cannot be used to prove $r \parallel s$?

(a) If two coplanar lines and a transversal form a pair of corresponding angles that are congruent, the lines are parallel.

(b) If two lines are perpendicular to the same line, they are parallel.

(c) If two coplanar lines and a transversal form a pair of consecutive interior angles that are supplementary, the lines are parallel.

(d) In a plane, if two lines are perpendicular to the same line, the lines are parallel.



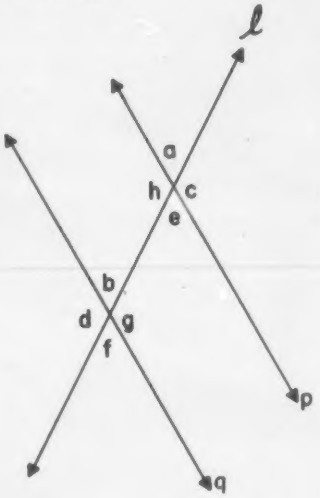
5. Given lines p and q and transversal ℓ, which of the following statements is false?

(a) If $m\angle a = 63$ and $m\angle b = 63$, then $p \parallel q$.

(b) If $m\angle c = 100$ and $m\angle g = 90$, then p is not parallel to q.

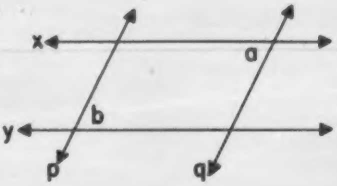
(c) If $m\angle d = 103$ and $m\angle a = 67$, then p is parallel to q.

(d) If $m\angle b = 56$ and $m\angle h = 124$, then $p \parallel q$.

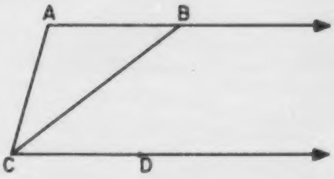


V. Write a proof for each of the following problems:

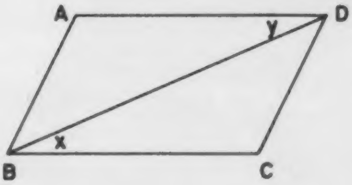
1. If two parallel lines are cut by a transversal, the lines which bisect a pair of corresponding angles are parallel.
2. If each pair of opposite sides of a quadrilateral are congruent, the quadrilateral is a parallelogram.
3. In the figure to the right, $x \parallel y$ and $p \parallel q$. Prove $m\angle a = m\angle b$.



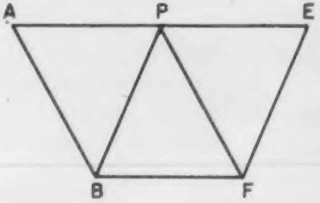
4. In the figure to the right, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, $AB = AC$. Prove $\overrightarrow{CB}$ is a midray of $\angle ACD$.



5. In the figure to the right, ABCD is a quadrilateral with $\overline{AB} \parallel \overline{CD}$ and $\angle x \cong \angle y$. Prove $\angle A \cong \angle C$.

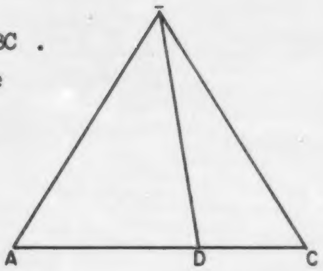


6. In the figure to the right, $\overline{PE} \parallel \overline{BF}$; $\overline{AP} \parallel \overline{BF}$. Prove: The points A, P, E are collinear.



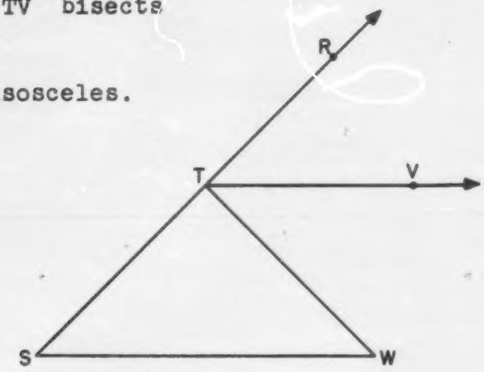
7. Given: $\triangle ABC$ with $AB = BC$. Points A, D, and C are collinear in that order.

   Prove: $AB > BD$.



8. Given: $\angle RTW$ is an exterior angle of $\triangle STW$. $\overrightarrow{TV}$ bisects $\angle RTW$. $\overleftrightarrow{TV} \parallel \overleftrightarrow{SW}$.

   Prove: $\triangle WST$ is isosceles.



Chapter 7

SIMILARITY

After introducing the notion of "same shape," we are naturally led to the need for considering proportionality. Our presentation is somewhat different from the usual ratio approach. It is more general and we believe it is simpler. It is our desire to use a definition that would avoid exceptional cases in the applications of the theorems of this chapter to the development of coordinate geometry that follows. This general approach should also be more useful to students in later mathematics. It is our belief that students should be aware of the number zero and should see that because of zero the ratio approach is quite limited.

However, for the sake of pedagogy, inasmuch as our work in this chapter will be dealing mainly with lengths of segments, we list special properties that are true of proportions involving only positive numbers. Before using any of these properties the students should always check to see whether the numbers involved are only positive numbers. If they might be zero or negative, then the student should rely on the definition of proportionality for his solutions.

Actually, there is no need to dwell at length on the complicated properties, nor to drill students in complicated algebraic maneuvers with proportions and ratios. Our uses of proportions follow easily and directly from the definition. In fact, using our notation, students are more likely to set up the corresponding numbers in a proportion correctly.

Our symbol for "are proportional to," is by no means universal. It is our own invention, analogous to many, as a symbol and as an invention. We do hope that it conveys the idea of an equivalence relation. The one-to-one correspondence is important in the definition of a proportionality; and you should find that as a result, the analogous definition of "inversely proportional," in Chapter 11, is much clearer than the usual interpretation.

Following the topic of proportion, a definition of similarity for polygons is stated, and using this we show that similarity has the reflexive, symmetric, and transitive properties. We then show how congruence is a special kind of similarity and can then define congruence for convex polygons. After this brief discussion our development concentrates on a study of similarities between triangles. We then use the Proportional Segments Postulate to establish that there actually exists a triangle similar to any given triangle with any given positive proportionality constant. This theorem (7-3) is not usually stated at this time. We placed it here because it simplifies the proofs of the basic similarity theorems for triangles. Two of the problems in the problem set immediately preceding it are intended to make its proof clearer to the student.

Our reason for postulating the basic similarity statement, the Proportional Segments Postulate, rather than attempting to prove it as a theorem is that on one hand, we feel it is intuitively reasonable, and on the other hand, by taking it as a postulate, we avoid the myriad difficulties inherent in discussing the statement in the incommensurable case.

We conclude the chapter with a study of altitudes of triangles, the projection of a point and a segment on a line, similarities between right triangles, and finally, with proofs of the Pythagorean Theorem and its converse.

The individual sections of this chapter make convenient teaching-units. We feel that the development presented clearly indicates that congruence is a special instance of similarity.

It may be appropriate after completing this chapter but before starting coordinates to take a few minutes to review some of the comments in Chapter 1 on postulational systems, now that the students have had some experience with one. They should have begun to see by this time how the system organizes geometry and have an appreciation of the distinction between physical geometry and a deductive system.

7-2 If your students have a knowledge of ratio and know that a statement, such as: "a is to b as c is to d" is equivalent to $\frac{a}{b} = \frac{c}{d}$ (provided $b \neq 0$, $d \neq 0$), then the teaching of our way of phrasing proportions should be based on this familiar background. For instance, suppose that the common ratio is 2 in the ratio stated, then the student should be shown that $\frac{a}{b} = \frac{c}{d} = 2$ is equivalent (provided $b \neq 0$, $d \neq 0$) to the two statements $a = 2b$ and $c = 2d$. With such an introduction, the student should see that the definitions just preceding Problem Set 7-2a merely express his familiar ideas about ratio in slightly different terminology. Certainly, the approach in the text is often encountered in the proofs of certain theorems about ratios.

The discussion in the text following Problem Set 7-2a concerns observations that are most conveniently made if the student has worked most of the problems. We recommend your looking at this discussion and thinking about your choice of problems for assignment with the discussion in mind.

7-3 Remind the students again that the vertices are numbered consecutively in such a way that $A_1 \longleftrightarrow B_1$, $A_2 \longleftrightarrow B_2$, ..., $A_n \longleftrightarrow B_n$. Certainly this is not the only way this can be written, but it is one of the most convenient.

Note the following in your examination of the five examples concerned with similarity:

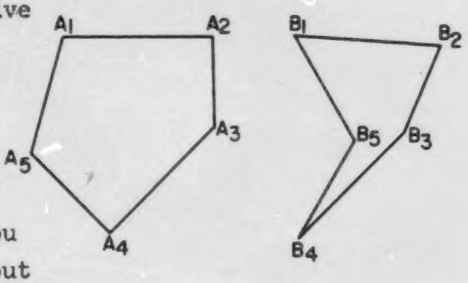
In the second example, the correspondence ABCD ↔ A'B'C'D' is not a similarity, since corresponding angles are not congruent.

In the third example, the correspondence PQRS ↔ UVWX is not a similarity, since corresponding sides are not proportional.

In the fourth example, the reason that the correspondence between EFGH and KLMN given by E ↔ K , F ↔ M , G ↔ L , H ↔ N cannot be a similarity, is that $\overline{EF}$ is a side of EFGH , but the corresponding segment $\overline{KM}$ is not a side of KLMN

In the fifth case considered, the identity correspondence ABC ↔ ABC is a similarity.

Another instance that illustrates the case where the measures of corresponding sides are proportional but the correspondence is not a similarity occurs when one of the polygons is not convex. Since we have not considered the concave polygon, in fact our definition specifies "convex," we did not describe this in the text, but it is a suitable illustration for you to use. As you talk about the concave figure, call attention to the definition of similarity again.



An appropriate word to fill the first blank is "congruent"; for the second blank, "congruence" is appropriate.

With regard to the use of the symbol, $\cong$, to denote "is congruent to," it can be pointed out that the symbol helps us to remember that congruence is the special instance of similarity, $\sim$, in which corresponding sides have equal, = , measures.

In our development we postulated congruence and proceeded from there to "similarity." It was pointed out in Talks how we might have reversed this procedure.

The definition here offers a splendid opportunity for the student to see that this might have been done. However, before you get to this page or when you are making the assignment, use questions to lead the student into making the connection between similarity and congruence himself. One approach might start "Are congruent triangles always similar?" and lead immediately to "Can you formulate another definition for congruence in terms of similarity?" From this a discussion of the alternate sequence could develop if desired and if time was available.

In studying Theorem 7-1, which shows that similarity is an equivalence relation, a set of cardboard models of similar polygons is helpful. Suppose that you have a pair such that a side of one is twice the corresponding side of the other. Certainly a side of the latter is one-half the corresponding side of the first polygon. A third polygon similar to one of these, but with a different proportionality constant, might be introduced to illustrate the transitive property.

For your own information, it might be helpful to reread the Talk entitled, Equality, Congruence, Equivalence.

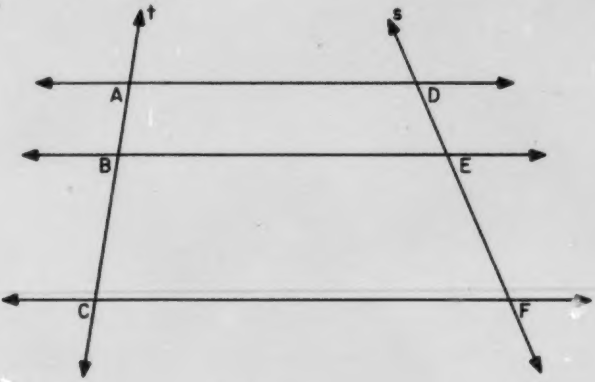
It is clear from our willingness to say "corresponding sides are proportional," rather than insisting on more precise phrasing, that we are not adverse to colloquial or semi-technical language, provided the student is mature enough to appreciate that the language is not precise and is able, on request, to speak or write with exactness and precision.

In the problem sets we have adopted the following convention. If we write $\triangle ABC \sim \triangle XYZ$ with a proportionality constant k , we are implying that the sides of the first named triangle are proportional to the sides of the second named triangle with the proportionality constant k ; that is, $AB = k \cdot XY$, $AC = k \cdot XZ$, and $BC = k \cdot YZ$.

7-4 In the experiment preceding Postulate 21, the teacher might suggest the use of lined paper to make the postulate more plausible. Let $\overline{BC}$, $\overline{DE}$ and A be on the lines of the paper.

In the proof of Theorem 7-2, it is tacitly assumed that if B is between A and C , then E is between D and F . A proof that this is the case can be based on the following theorem known as the Parallel Projection Theorem.

THEOREM. Given two transversals t and s intersecting three parallel lines $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$, $\overleftrightarrow{CF}$ in points A, B and C , and D, E and F , respectively. If B is between A and C , then E is between D and F .



Proof: Since $\overleftrightarrow{AD} \parallel \overleftrightarrow{BE}$, then the segment $\overline{AD}$ cannot intersect $\overleftrightarrow{BE}$, and hence, A and D are on the same side of $\overleftrightarrow{BE}$. Likewise, since $\overleftrightarrow{CF} \parallel \overleftrightarrow{BE}$, then the segment $\overline{CF}$ cannot intersect $\overleftrightarrow{BE}$, and C and F are on the same side of $\overleftrightarrow{BE}$. Since B is between A and C by hypothesis, segment $\overline{AC}$ intersects $\overleftrightarrow{BE}$ at B ; hence, A and C are on opposite sides of $\overleftrightarrow{BE}$. Since D and A are in the same halfplane determined by $\overleftrightarrow{BE}$, and F and C are in the same halfplane, and A and C are in opposite halfplanes, then it follows that D and F are in opposite halfplanes determined by $\overleftrightarrow{BE}$. Hence, $\overline{DF}$ meets $\overleftrightarrow{BE}$ in a point which must be E , since E is the intersection of $\overleftrightarrow{DF}$ and $\overleftrightarrow{BE}$. Therefore, E is between D and F . We have assumed that $A \neq D$ and $C \neq F$.

The argument is easily modified to apply to the cases where $A = D$ or $C = F$.

7-5 The Existence Theorem, Theorem 7-3, simplifies the proofs of S.S.S., S.A.S., and A.A. Similarity Theorems. Its proof should be easy to follow if students have done Problems 9 and 10 in Problem Set 7-4.

Here again we did not think it advisable to take time to point out to the students in the text how we know that point E lies between A and C . The proof does follow quite easily, however, from the Plane Separation Theorem, which can be found in this Teachers' Commentary in Comments on Chapter 4. It says that if a line intersects one side of a triangle in an interior point and does not contain the opposite vertex, then it must intersect one of the other two sides in an interior point. D is in the interior of $\overline{AB}$, and $\overleftrightarrow{DE}$ does not contain C or any other point of $\overline{BC}$ since it is parallel to it. Therefore, $\overleftrightarrow{DE}$ must intersect $\overline{AC}$, the only remaining side of the triangle, in an interior point. E is the intersection of $\overleftrightarrow{AC}$ and $\overleftrightarrow{DE}$, so E is between A and C .

7-6 There is a slight departure here from our previous use of the word "determine." Prior to this we gave examples such as "Two points determine a line," which implied that there was a unique line which contains the two points. We also said that two segments might determine an angle; here, again, there would be a unique angle which would contain the segments. From this, the students may have the impression that "determine" implies the idea "contain." However, this is not always the case. Certainly here, when we say a vertex P and the opposite side $\overline{QR}$ determine an altitude, we do not mean that the altitude contains the side $\overline{QR}$. Rather, we use the word "determine" to say that certain conditions specify a unique set, particularly a unique set of points, a geometric figure. A set of points is determined by certain given conditions if and only if there is one and only one such set that satisfies those conditions.

Another idea related to perpendicularity is projection. It is introduced not only because of its importance in later mathematics, but also because it helps us to better phrase Corollaries 7-7-1 and 7-7-2. Later, in Chapter 9, definitions will be given for the projection of a point into a plane.

Notice that the projection of a given segment, $\overline{AB}$, into a line, ℓ, is a segment unless $\overline{AB}$ is contained in a plane which is perpendicular to the given line. For example, if $\overleftrightarrow{AB}$ is perpendicular to ℓ, then the projection of $\overline{AB}$ is a single point.

7-7 Interest in the Pythagorean relations is reflected by the abundance of current literature, pamphlets and periodical articles that treat this subject. More proofs (over 200) have been found for this theorem and its converse than for any other. Reviewing recent indexes of The Mathematics Teacher will uncover a number of articles of interest which treat this subject.

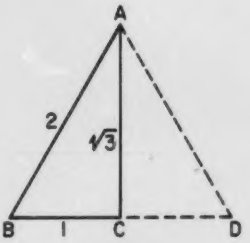
A relationship that is interesting to students and helpful to teachers in developing quiz problems, is an algebraic interpretation of the lengths of the sides. Let x and y be relatively prime (i.e., have no common prime factor) with $x > y$. Then $x^2 - y^2$, $2xy$, and $x^2 + y^2$

are sides of a right triangle. This follows, since $(x^2 - y^2)^2 + (2xy)^2 = (x^2 + y^2)^2$. Of course, there are other possibilities, e.g., $x - y$, $2\sqrt{xy}$, and $x + y$, but the first statement is easiest to work with.

In making up a test you may face the situation where you don't want to use one of the familiar combinations of sides of a right triangle where some student might get the answer by remembering it. The above formula is a crank-turning affair that will help you produce the needed test questions. For example, let $x = 5$ and $y = 2$; we have a triangle whose sides are 29 , 21 , and 20 .

7-8 In the discussion of the 30-60-90 triangles you might prefer this alternate proof that if the sides of a triangle are proportional to 1 , $\sqrt{3}$, 2 , then the triangle is a right triangle with acute angles measuring 30 and 60 .

Proof: Given $\triangle ABC$, with $(BC, CA, AB) \underset{p}{=} (1, \sqrt{3}, 2)$ Then since $k^2 + (k\sqrt{3})^2 = (2k)^2$, we know $(BC)^2 + (CA)^2 = (AB)^2$. Thus $\triangle ABC$ is a right triangle with right angle at C . Consider D in $\overrightarrow{BC}$ so that C is between B and D and $BC = CD$, and thus $BD = 2$. We can show that $\triangle BAC \cong \triangle DAC$ by S.A.S. Therefore $AD = AB = 2$ and the triangle ABD is equilateral. Thus $m\angle B = 60$ and $m\angle BAC = 30$.



The results of Theorem 7-10 can be used as a rapid means of computing the other two sides of a 30-60 triangle given any one of the sides. We have not drilled on special formulas involving only two sides of these special right triangles since we thought that using the proportionalities was more in keeping with the development presented here. However, in Chapter 11, in the work with area it will probably be very convenient if the student can readily obtain the side opposite the 30, 45, or 60 degree angle having been given the hypotenuse. At that time we review the relations which exist and suggest the following formulas where c represents the length of the hypotenuse and a the side opposite $\angle A$.

1. If $m \angle A = 30$, $a = \frac{1}{2} c$.
2. If $m \angle A = 45$, $a = \frac{1}{2} c \sqrt{2}$.
3. If $m \angle A = 60$, $a = \frac{1}{2} c \sqrt{3}$.

Illustrative Text Items for Chapter 7

1. Find x in each of the following proportions:

   (a) $(3,7) \underset{p}{=} (x,77)$.

   (b) $(3,x) \underset{p}{=} (x,27)$, if $x > 0$.

   (c) $(2,x) \underset{p}{=} (x,1)$, if $x > 0$.

   (d) $(3,0) \underset{p}{=} (6,x)$.

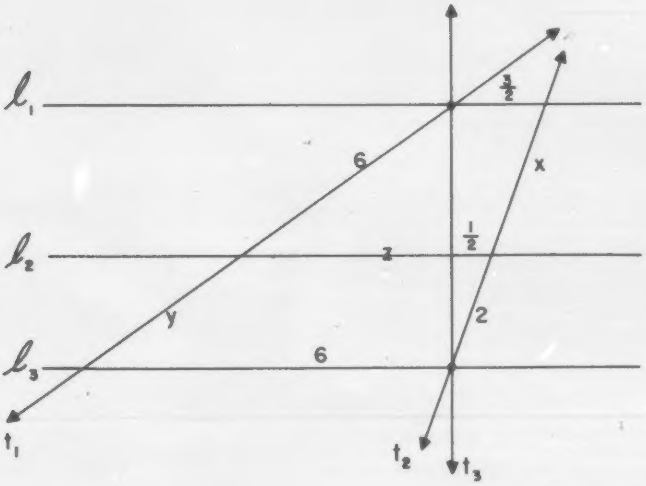
   (e) $(3,x) \underset{p}{=} (6,x)$.

   (f) $(\sqrt{3}, 2) \underset{p}{=} (6,x)$.

   (g) $(-2, x) \underset{p}{=} (5,7)$.

2. In the following figure $\ell_1 \parallel \ell_2 \parallel \ell_3$, and t_1 , t_2 , t_3 are transversals. The measures of certain segments are given.

   Find x, y and z .


ℓ_1
$\frac{3}{2}$
6
x
ℓ_2
$\frac{1}{2}$
z
y
2
ℓ_3
6
t_1
t_2
t_3

3. Listed below are the lengths of the three sides of eight triangles. (1) List the pairs of similar triangles. (2) List the right triangles.

(a)	6 ,	8 ,	11
(b)	5 ,	13 ,	12
(c)	17 ,	6 ,	15
(d)	24 ,	25 ,	7
(e)	24 ,	33 ,	18
(f)	$2\sqrt{2}$,	$2\sqrt{2}$,	2
(g)	4 ,	5 ,	6
(h)	4 ,	$2\sqrt{2}$,	4

4. $\triangle ABC \sim \triangle RST$ with proportionality constant 2 and $\triangle RST \sim \triangle XYZ$ with proportionality constant 6 . Why is $\triangle ABC \sim \triangle XYZ$? Give the proportionality constant for this last similarity.

5. Given $\triangle ABC$ a right triangle with $\overline{CD}$ the altitude to the hypotenuse. $CD = \sqrt{3}$. $AD = 1$. Find DB, AC, BC and AB .

6. Given $\triangle ABC$ an isosceles triangle with $AB = AC = 3\sqrt{2}$ and $BC = 6$. Find $m \angle ABC$.

7. The measure of the altitude of an equilateral triangle is 10 . Find the perimeter of the triangle.

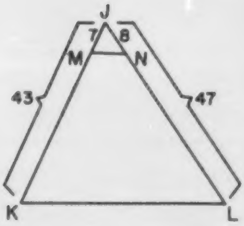
8. One side of an equilateral triangle has a length of 6 . What is the length of each altitude?

9. One angle of a right triangle has a measure of 60 . If the shortest side of the triangle measures 8 inches, what is the measure of the hypotenuse?

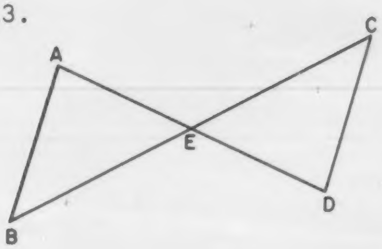
10. In $\triangle ABC$, $m \angle C = 90$, $AB = 17$, $AC = 15$. Find BC .

11. The sides of a triangle measure 6, 9, 12 . Find the perimeter of a similar triangle whose longest side is 8 .

12. In $\triangle JKL$, $JM = 7$, $JK = 43$, $JN = 8$, $JL = 47$. Is $\overleftrightarrow{MN} \parallel \overleftrightarrow{KL}$, or does $\overleftrightarrow{MN}$ intersect $\overleftrightarrow{KL}$?



13.

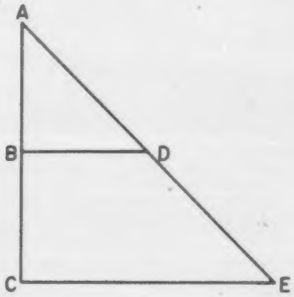


$\overline{AD}$ intersects $\overline{CB}$ at E . $(AE, BE) \underset{p}{=} (EC, ED)$.

$\triangle ABE$ is not isosceles.

Prove that $\angle B$ is congruent to some angle in the figure.

14.



Hypothesis: $\overline{BD} \parallel \overline{CE}$,
B is the midpoint of $\overline{AC}$.

Prove: $BD = \frac{1}{2} CE$.

15. The numbers a, b, c, d are positive. $(a,b) \underset{p}{=} (c,d)$. Which of the following are true? Give a reason for your answer.

(a) $(b,a) \underset{p}{=} (c,d)$.

(b) $(a,c) \underset{p}{=} (b,d)$.

(c) $a = kc$, $b = k \cdot d$.

(d) $ac = bd$.

(e) $bc = ad$.

(f) $(b,a) \underset{p}{=} (d,c)$.

16. State the S.A.S. Similarity Theorem.

FACTS AND THEORIES

Science today is playing an increasingly important part in the life of the individual. No one can claim to be truly educated unless he has a reasonable understanding of the facts and methods of science. This does not mean that we must all become nuclear physicists, nor that we must spend all our time reading books and attending lectures on the latest collection of particles discovered by the physicists. But it does impose on us the obligation to learn enough of the facts of modern science to provide a foundation for understanding. It does imply an intelligent selection of material to be learned.

We, as educators, are expecially obligated to make such a selection for our students. They come to us with a miscellaneous hodgepodge of disjointed facts and pseudo-facts, gleaned from newspapers, magazines, books, and other sources. We must help them--with our own limited information--to straighten out their ideas, to build a reasonable conceptual structure upon which they can hang new facts, to distinguish between that which is significant and that which is not, and, perhaps most important of all, to understand how new knowledge is acquired. If pursued to the extreme, this last goal would lead us to the far reaches of epistomology and scientific method, which have been the subjects of many weighty tomes written by scholars over many lifetimes, and about which the last word has certainly not been uttered. But to dismiss this topic entirely as being too subtle for the immature minds of our students is to deny them the opportunity of becoming a little more mature in our classrooms.

What should be the aims of the mathematics teacher, in the light of what we have just said?

Certainly we should help the student to become acquainted with the facts of mathematics by working with them. We agree that our subject is an essential tool in science and in daily life, and that the student should acquire a working facility in it. Therefore we teach him arithmetic, elementary algebra, intuitive geometry in the lower grades, advanced algebra, synthetic and analytic geometry, possibly calculus and other topics in the higher grades.

It would be difficult, however, to defend the teaching of all these subjects on the grounds of utility alone. No one pretends, for example, that it is of practical importance that the bisector of an angle of a triangle divides the opposite side in the way that it does. We proceed, then, to the second aim, of developing in the student an appreciation of clear, logical reasoning as exemplified in mathematics, and an ability to transfer this type of reasoning to other situations. We have been moderately, though not eminently, successful in this respect in the past. Whether our present efforts will tend to further this objective remains to be seen. We certainly hope so.

A third aim, which has been receiving more attention of late, is to develop in the student an understanding of the structure of mathematical systems. We are beginning to speak of closure, commutativity, distributivity and so on in dealing with number systems, and--still too timidly, perhaps--of the axiomatic nature of geometry.

This third aim is closely related to the broader one mentioned earlier, of helping the student to understand how new knowledge is acquired, how man learns about the physical world, how he constructs, develops and tests theories about the physical, biological, social, and economic aspects of life around him. Let us address ourselves briefly to these questions.

Whether we recognize it or not, theory plays an indispensable role in our study of any field whatsoever. The acts of naming, classifying, and generalizing are conceptual in nature. Even emotional reactions to stimuli depend on a structuring of experience. This structuring may be based on analogies that we recognize, on inductive reasoning, or on a combination of the two.

One of the great intellectual experiences in any person's life occurs at the moment that he realizes that there can be a distinction between the real world--whatever that may mean--and the conceptual world. Our great debt to Euclid is that his mode of organization of geometry, using postulates and axioms from which theorems were deduced, has made this distinction between concepts and the things by which they were suggested a part of our intellectual heritage. One of the major factors in the ever accelerating growth of mathematics and science today is that, from the act of recognition of the distinction we have been discussing, we have evolved in our thinking to a stage at which we are able to use this distinction as a research tool. Indeed, we are able to recognize consciously that the conceptual world at one stage may be used as raw material for a theory at a later stage. Thus, we are not always forced to refer back to the primary data suggested by our senses. For example, the classical geometry of various surfaces in three dimensions may be taken as the jumping off place for a study of metric spaces by comparing it with that of classical theory.

In every case, then, we operate simultaneously at two different "levels." One is the primary, intuitive level, containing the raw data from which our theory will be abstracted. Following some authors, we call this primitive intuitive level the "P-level." The second level is conceptual, denoted the "C-level." Initially, the C-level is empty, waiting to be filled with the concepts and relations that we construct.

We have complete freedom with respect to the concepts and relations which we choose to insert in the C-level, so long as we do not assert any connection between it and the P-level. Naturally, we hope eventually to set up a correspondence between the two levels, and this hope guides our constructions and our choice of language. Logically, there is no necessity to make the language in the C-level correspond to that of the

P-level, and in order to avoid confusion it might be better to use different terms entirely. For example, the "points," "lines," and "planes" of axiomatic or postulational geometry (the C-level) might be replaced by other terms which have not been preempted in physical geometry (P-level). But once the formal distinction between the two levels and their languages has been established and understood, there is a psychological advantage to be gained from the use of the same terms, for the proposed correspondence is then transparently indicated. Thus, we know that the geometrical "point" is meant to correspond to the physical point, the geometrical "line" to the physical line, and so on. We can intuit, conjecture, and then perhaps prove theorems at the C-level by peeking over into the P-level at the corresponding "facts," arrived at by experiment there. For example, the concurrence of the medians of a triangle could be guessed from drawing a number of physical triangles and their medians on a piece of paper. This type of experience is extremely valuable and constitutes an important psychological adjunct to mathematical discovery. It must be pointed out carefully, though, that formal proof by deductive reasoning from postulates at the C-level is necessary. Furthermore, the logical conclusion to be drawn from this combined guessing and proving process is not that we have made the geometrical theorem more certain by experimental verification. The truth of the theorem has been established (at the C-level) with complete certainty by logical deduction from the axioms. What we tend to establish by empirical tests is the adequacy of our postulate system to bring about a close correspondence between the C-level and P-level. We feel satisfaction if the experiments tend to confirm the correspondence we had in mind.

Often, by comparing one theory with another, we obtain useful information not otherwise available. Consider, for example, what our situation would be if we start at a certain P-level (physical geometry), and construct two different C-levels. One C-level contains all the postulates of Euclidean geometry except the parallel postulate which is replaced by a postulate asserting the existence of more than one parallel. This C-level we call a non-Euclidean geometry. We find that if one of these systems contains an inconsistency or flaw in its development then the other does also. Furthermore, we find that it is a moot point as to which of the geometries provides a better description of physical space--i.e., as to which of the C-levels is in closer correspondence with the P-level. Indeed, the investigation of this very question led to our present day deeper understanding of the connection between fact and theory.

What are the considerations that govern our choice of undefined elements and relations and unproved propositions (axioms, postulates)? Certainly we want our system to be consistent: a proposition and its contradiction should not both be provable in the system. If we regard our axioms as inputs and our theorems as outputs, then economy and fruitfulness are desirable as increasing output per unit input. Of course, this analogy is not to be taken too seriously, but it indicates why we should not postulate everything. Unfortunately some geometry texts nowadays go to the extreme of setting down fifty or more postulates. There is nothing logically wrong with this, but it militates against economy, elegance, intuitiveness, simplicity, and ease of verification in a particular interpretation--properties that are certainly desirable.

One property that we have not mentioned is that of being categorical. This means that every two concrete interpretations (models) of the system are essentially the same: it is possible to set up a one-to-one correspondence between the elements and relations of the two interpretations, so that they may be regarded as identical except for the names assigned to the elements and relations. The two models are then said to be isomorphic. If we start with a particular P-level and wish to describe it completely by means of an axiom system, without permitting any non-isomorphic models, then we try to make our system categorical. This is the case with Euclidean geometry or the real number system.

Sometimes we reap an unexpected harvest from the construction of a categorical system. We may find two apparently different interpretations, and can then conclude that they are essentially identical because the system is categorical. Any theorem which holds in one model is then sure to hold in the other. An example is the pair of models M_1, consisting of the real numbers under addition, and M_2, consisting of the positive real numbers under multiplication. The one-to-one correspondence $M_1 \longleftrightarrow M_2$ is established by the exponential function (from M_1 to M_2) and the logarithm from M_2 to M_1). Another example is the pair of physical processes, diffusion of a gas and heat-flow, both being governed by the same differential equation. Still another example is the isomorphism of Euclidean plane geometry with the collection of all real-number pairs. This isomorphism allows us to solve geometrical problems by means of algebra, and vice versa.

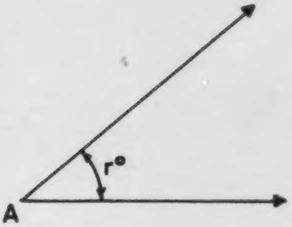
At other times, we find it more profitable to make our system non-categorical. This is true when we have several P-levels which bear some resemblance to each other. If we can construct a suitable C-level so that each of the P-levels is an interpretation of it, then anything we prove in the C-level will hold in all of its non-isomorphic models. This happens, for example, in the case of group theory. It also happens when we state some, but not all of the axioms of Euclidean geometry. In this case our theorems, being provable, say, without the parallel postulate, must hold also for all geometries satisfying the stated axioms. There is no reason to hide this desirable state of affairs from our students, for fear of violating their intuitions about space. Rather, we should regard such occasions as valuable opportunities for teaching an important lesson.

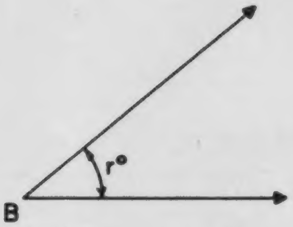
Our discussion here has been far from exhaustive. We hope that it has served the purpose of pointing to a desirable and sometimes neglected goal in education, and that it has indicated how we, as teachers of mathematics, can approach this goal.

EQUALITY, CONGRUENCE, AND EQUIVALENCE

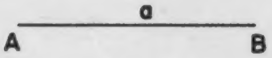
1. Angles and Segments.

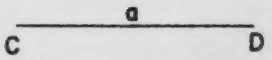
In describing the relation of "equality" between angles and segments, this book departs from what has been common usage. Before explaining why this has been done, let us first note quickly how the new usage compares with the old. Suppose we have two angles with the same degree measure r, like this.





and two segments of the same length, like this:





In these two instances, the facts are plain. They would be reported in the following ways, in the old and new terminologies.

Old	New	Old	New
The angles are equal.	The angles are congruent (or they have the same measure).	$\angle A = \angle B$	$\angle A = \angle B$ (or m $\angle A = \angle B$)
The segments are equal.	The segments are congruent (or they have the same measure.	$AB = CD$	$\overline{AB} \cong \overline{CD}$ (or $AB = CD$).

From the table it is plain that the new usage is not complicated. We have simply substituted one word for another, and one symbol for another. Of course, even simple changes should be made only for good reasons; they go against everybody's habits, and cause more trouble at first than their simplicity would suggest. We believe that there are good reasons for the use that we have made of the word congruence. Following is an explanation of what these reasons are.

2. Various Kinds of Equality.

The word "equals" is commonly used in mathematics in at least this many different senses:

(1) When we write $2 + 4 = 3 + 3$, we mean that the number denoted by $2 + 4$ and the number denoted by $3 + 3$ are exactly the same number (namely, 6). Here "equals" means "is the same as."

(2) When we say that two angles are equal, we mean that they have the same measure, or the same shape.

(3) Two circles are equal if and only if they have the same radius.

(4) Two segments are equal if and only if they have the same length.

(5) Two triangles are equal if and only if they have the same area.

(6) Two polyhedrons are equal if and only if they have the same volume.

These uses of "equals" divide sharply into three groups.

(I) The first meaning ("is the same as") stands entirely alone. This is the logical identity. It arises in all branches of mathematics, including geometry. This is the use in (1).

(II) "Equality" expresses the same basic idea for angles, circles, and segments, in (2), (3), and (4). It means in each case that the first figure is the same size and shape as the second. (It can also be explained in terms of rigid motion. For an explanation, see the Appendix on Rigid Motion in Part III of the text.) This idea of same size and shape is geometric, and is one of the most basic ideas in geometry. Applied to triangles, it is always described as congruence and not as equality.

(III) "Equality" to mean equal areas or equal volumes, as in (5) and (6), implies that two things are equal if and only if they contain the same amount of "stuff."

These are the three main ideas involved. We notice that the words and the ideas overlap both ways. Not only is the word "equals" used in widely different senses as between (II) and (III), but the basic idea expressed in (II), that is, involved in (2), (3), and (4) is expressed by apparently unrelated words.

Obviously students can and do learn to keep track of what is meant, even when the words and the ideas overlap in this way. All of us learned to do this, when we were in the tenth grade. The whole thing becomes easier to learn, however, and easier to keep track of, if the words match up with the ideas in a simpler and more natural way. This can be done as follows:

(I) We can agree to write "=", and say "equals," if and only if we mean "is the same as." (This is the standard usage in nearly all of modern mathematics.)

(II) We already have a word to express the idea that one triangle can be made to "coincide" with another; we say that they are congruent. We can use the same word to express the same idea when we are talking about angles, circles or segments.

(III) When we want to convey the idea of equality of area or of volume, we say so explicitly.

Notice that by doing this we have not introduced any new words into the language of geometry. We are not trying to complicate matters. All that we are trying to achieve is a situation in which the familiar and available words correspond in a natural way to the familiar and basic ideas. The correspondence looks like this:

(I) $=$, between any two geometric figures whatever, means "is the same as."

(II) $\cong$, between any two geometric figures whatever, means that one can be "moved" so as to coincide with the other.

(III) Equality of area, equality of volume, and so on, are to be described explicitly as such.

All this is straightforward language. We believe that your students will find it easy to learn and easy to use.

3. Equivalence Relations.

All the uses of "equals," in mathematics or otherwise, involve the notion of two things being *alike in some respect*. The particular respect to be considered may be made explicit, as in usage (5) on page 10, or it may not, as in "All men are created equal." As mentioned previously, mathematicians have pretty generally agreed to use the word to mean "alike in *all* respects"; that is, identical. Instead of the other usage they speak of an "equivalence relation." A relation between pairs of objects, from some given set, is called an *equivalence relation* if it has the following three properties:

(1) It is reflexive. That is, any object of the set is equivalent to itself.

(2) It is symmetric. That is, if A is equivalent to B, then B is equivalent to A.

(3) It is transitive. That is, if A is equivalent to B, and B is equivalent to C, then A is equivalent to C.

In mathematical development we may use several different kinds of equivalence relations. To keep them separate we give them different names and different symbols. In our geometry we have used the following equivalence relations.

(a) Identity. The relation "is the same as" is easily seen to satisfy the three properties listed on the previous page. The word "equal" and the symbol "=" are reserved for this equivalence relation.

(b) Congruence. Here again, the properties are easily checked. (Refer to the Talk on Congruence for a general treatment.) The symbol is "$\cong$".

(c) Similarity. Here again we have an equivalence relation, denoted by "$\sim$".

(d) We have not introduced any special notation for "equality of area," or "equality of volume," but each of these relations is reflexive, symmetric and transitive. We could, if it were convenient, introduce words and symbols for these equivalence relations.

Insistence on exactitude of language and symbolism is characteristic of modern mathematics.

4. Classification and Equivalence Classes.

Equivalence relations are connected closely with the idea of classification, another important concept in mathematics.

We are familiar in our every day life with the process of sorting or classifying the objects in a collection or set. For instance, a teacher may sort a collection of examination papers into several piles, each pile consisting of all those papers with a certain grade. While this method of sorting might be convenient for some purposes, it might be convenient for other purposes to sort the papers into piles such that two papers are in the same pile if and only if the students' last name as indicated on the papers begin with the same letter. In both methods there is a definite rule for sorting. This rule was such that no matter which paper was considered it was clear that it belonged to a pile and further that there was only one pile to which it belonged.

No matter which method of sorting papers was used, we see that two papers from the same pile are alike or equivalent with respect to some property, and that two papers from different piles are not alike or equivalent with respect to that property. We see, in fact, that in both methods of sorting we have an equivalence relation between pairs of examination papers, for in each method the given relation is reflexive, symmetric, and transitive.

From these examples it is easy to see how one can imagine using an equivalence relation between pairs of objects in a given set to sort the objects into piles. Indeed, the "sorting rule" is that two objects are in the same pile if and only if they are equivalent. It is usual to call the subsets, or piles, thus obtained equivalence classes.

If we apply these ideas to the equivalence relation of congruence between pairs of triangles, we see that we can think of congruence as sorting or classifying triangles having the same size and shape. Each equivalence class consists of all triangles having the same size and shape.

Let us reconsider certain aspects of our examples concerning the sorting of examination papers. Suppose the unlikely event occurred that every pair of papers that had the same grade were written by students whose last names began with the same letter. If in addition, every pair of papers written by the students' whose last names began with the same letter also had the same grade, then the classification by either one of the methods of sorting would produce exactly the same piles or equivalence classes. The lesson to be learned from observing this situation is that, although every equivalence relation on a given set can be used to sort the elements of the set into equivalence classes, it may not be possible from inspection of the piles to determine the equivalence relation that led to them.

Now suppose that we are given a collection of subsets of a given set, such that the union of these is the given set and, such that no two of them have an element in common. It is then clear that there is at least one equivalence relation for which the given subsets are equivalence classes. For instance, we could define two objects in the set to be equivalent if and only if they are in the same subset. This is an equivalence relation and its equivalence classes are the given subsets.

In summary of these general observations, we see that:

(1) from every equivalence relation we can obtain a related notion of classification.

(2) from a classification we cannot necessarily recover the original definition of the equivalence relation to which it is related; but

(3) for any classification there does not exist an equivalence relation from which the classification can be derived.

In geometry only Item (1) is important since we do not start with classifications and encounter the problem of determining appropriate equivalence relations. Instead, we start with equivalence relations and use the resulting classification.

THE CONCEPT OF CONGRUENCE

Congruence is a rich and complex idea with many ramifications in geometry--there really is nothing quite like it in algebra. It applies to figures of all kinds--segments, angles, triangles, circular arcs, polygons, truncated pyramids--in face to any conceivable figure. It plays an essential role in the theory of geometric measure of length, area and volume--it is intimately related to the important concept of rigid motion.

In this talk we examine carefully the conventional theory of congruence and the related theory of linear measure. This theory, as we present it in Part I below, follows generally the pattern established by Euclid. In Part II this conventional theory is contrasted with the theory of congruence in our G. W. text. You may feel that our discussion in Part II is somewhat condensed. But we believe the discussion is detailed enough for our purpose here. Finally, in Part III we discuss the concept of congruence for general figures and its relation to the idea of rigid motion.

Part I. The Conventional Theory of Congruence and Linear Measure

I-1. *Congruence in terms of size and shape*. The term congruence immediately calls to mind the famous dictum: Two figures are congruent if they have the same size and the same shape. Certainly this statement emphasizes the basic intuitive or informal idea that if two figures are congruent, one is a "replica" of the other. Also it points up the important property that if we know two figures to be congruent we can infer that they have the same area (or volume) and that they are similar.

But this is not the essential issue. It is: Does our dictum define congruence? Is it really a formal definition of the term congruence in terms of more basic ideas? Clearly the answer is no. For the notions size and shape are more complex than congruence. In order to measure (or define) size (area or volume) we try to find out how many congruent replicas of a basic figure (for example, square or cube) "fill out" a given figure. So actually it would be more natural and simple to base the theory of size (and shape) on the idea of congruence rather than the reverse.

I-2. Congruence in terms of rigid motion. But there are other "definitions" of congruence which we must discuss--consider the famous, "Two figures are congruent if they can be made to coincide by a rigid motion." Let us analyze this. Conceived concretely, say in terms of two paper heart-shaped valentines, it affords an excellent illustration of the intuitive idea of congruence and emphasizes again that one is a "replica" of the other. But this illustration, like most physical situations, does not have the precision required for an abstract mathematical concept. Surely we would have to pick up the first valentine and move it with almost infinite gentleness to prevent bending it slightly when getting it to coincide with the second one. And how could we be certain of perfect coincidence of the two valentines? Wouldn't this require perfect eyesight? It is clear that this "definition" interpreted concretely gives us a physical approximation to the abstract idea of congruence but doesn't define it. Moreover it is not even applicable in many physical situations: you hardly could get two "congruent" billiard balls to coincide by a rigid motion.

Should we then conclude that the idea of rigid motion is essentially physical and cannot be mathematicized as an abstract geometrical concept? Definitely not. Some mathematicians are ingenious and clever people and it might be a mistake to decide beforehand that they could not construct a precise abstraction from a given physical idea. Most familiar mathematical abstractions had their origin in concrete physical situations--certainly geometry had its origin in practical problems of surveying the heavens and the earth.

Let us table for the present the question of whether we can form an abstract geometrical theory of rigid motions. It would seem that a treatment of congruence based on a logically satisfactory theory of rigid motion could not be elementary and would hardly be suitable for a first course. In any case, without deeper analysis, the second "definition" is not a definition at all and might more properly be considered a statement of a property which rigid motions should have: namely, that any rigid motion transforms a figure into a congruent one.

I-3. Another definition. Consider and criticize a third suggested "definition": Two (plane) figures are congruent if a copy of the first made on tracing paper can be made to coincide with the second.

I-4. Congruence of segments. Since our three "definitions" do not define congruence we must probe more deeply. Here, as so often in solving problems, the imperialist maxim, "Divide and conquer", is very helpful. Instead of tackling the concept of congruence in its most complex form, that is for arbitrary figures, let us begin by considering a simple special case. A line segment--or as we shall call it--a segment is one of the simplest and most important geometric figures. We naturally begin by considering congruence of segments.

Let us recall how this is treated in Euclid or in the conventional high school geometry course. Congruent segments, usually called equal segments, are conceived as "replicas" of each other, in general with different locations in space. Congruent segments may coincide or be identical but they don't

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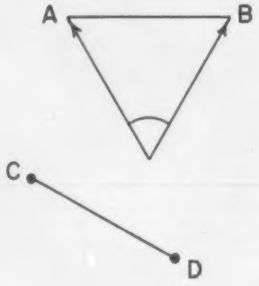
have to. If segments $\overline{AB}$ and $\overline{CD}$ are congruent we may interpret this concretely to mean $\overline{AB}$ and $\overline{CD}$ are "caliper equivalent"--that is if a pair of calipers is set so that the ends coincide with A and B then without changing the setting, the ends of the calipers can be made to coincide with C and D.



I-5. Basic properties of congruence of segments. What is the logical significance of congruence of segments in Euclid? Actually it is taken to be an undefined term. More precisely, using the notation $\overline{AB} \cong \overline{CD}$, congruence is a basic relation between the segments $\overline{AB}$ and $\overline{CD}$ which we do not attempt to define. We study it (as always in mathematics) in terms of its basic properties which are formally stated as postulates. Some of these postulates, which are not explicit in Euclid nor in most geometry texts are:

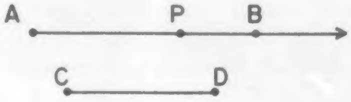
(1) (Reflexive Law) $\overline{AB} \cong \overline{AB}$;

(2) (Symmetry Law) If $\overline{AB} \cong \overline{CD}$ then $\overline{CD} \cong \overline{AB}$;

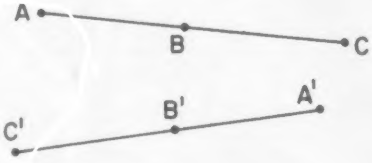
(3) (Transitive Law) If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$ then $\overline{AB} \cong \overline{EF}$.

That is, congruence of segments satisfies the three basic properties of equality or identity and so is an example of an equivalence relation. We must not assume that congruence means identity, since distinct segments can be congruent.

(4) (Location Postulate) Let $\overrightarrow{AB}$ be a ray and let $\overline{CD}$ be a segment. Then there exists a unique point P in $\overrightarrow{AB}$ such that $\overline{AP} \cong \overline{CD}$.



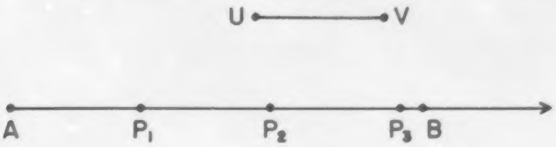
(5) (Additivity Postulate) Suppose $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$, B is between A and C and B' is between A' and C'. Then $\overline{AC} \cong \overline{A'C'}$.



We insert a few words on the important mathematical idea of equivalence relation. The most basic example of an equivalence relation and the one which suggests the concept is the relation equality or identity. Equivalence relations abound in geometry; for a discussion of them see the Talk on Equality, Congruence, and Equivalence.

I-6. Theory of linear measure. Segments are geometric figures, not numbers. But they can be measured by numbers--they do have lengths. In the conventional high school treatment it is assumed with little discussion that lengths of segments can be defined as real numbers. We indicate how to do this. Although the result is familiar, the process is complex and subtle and requires for its complete justification additional postulates. However, Postulates (1), ..., (5) above are sufficient for an understanding of the process.

We begin by choosing a segment $\overline{UV}$ which will be unchanged throughout the discussion (a so-called "unit" segment). Now given any segment $\overline{AB}$ we want to measure $\overline{AB}$ in terms of $\overline{UV}$. This involves a "laying-off" process. We take the ray $\overrightarrow{AB}$ and lay-off $\overline{UV}$ on it repeatedly, starting at A. Speaking precisely, there is a point P_1 in $\overrightarrow{AB}$ such that $\overline{UV} \cong \overline{AP_1}$. Similarly, we can show that there is a point P_2 in $\overrightarrow{AB}$ such that (a) $\overline{UV} \cong \overline{P_1P_2}$ and (b) P_1 is between A and P_2. For convenience we write condition (b) as (AP_1P_2). Continuing, there is a point P_3 such that $\overline{UV} \cong \overline{P_2P_3}$ and $(P_1P_2P_3)$. By this process we develop a sequence of points P_1, P_2, ..., P_n, ... on $\overrightarrow{AB}$ such that



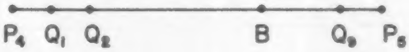
(1) $\overline{UV} \cong \overline{P_1P_2} \cong \overline{P_2P_3} \cong \ldots \cong \overline{P_{n-1}P_n}$, and

(2) (AP_1P_2), $(P_1P_2P_3)$, ..., $(P_{n-2}P_{n-1}P_n)$.

Intuitively (1) and (2) say that $\overline{UV}$ is laid-off on $\overline{AB}$ n times in a given direction--but note how very precisely and objectively (1), (2) say this, avoiding the somewhat vague terms "laying-off" and "direction." From another viewpoint we are laying the basis for a coordinate system on the line by locating precisely the points P_1, P_2, ..., P_n, ... which are to correspond to the integers 1, 2, ..., n, ...

Now what has this to do with the measure of $\overline{AB}$? Clearly we must learn how B is related to the points P_1, P_2, P_3, ... In the simplest case one of these might coincide with B, for example, $P_3 = B$. Then of course we define the measure of $\overline{AB}$ to be 3.

I-7. Refinement of the approximation process. You may ask, "Did we have to go through this elaborate process to explain that if the "unit" segment $\overline{UV}$ exactly covers $\overline{AB}$ three times, then the measure of $\overline{AB}$ is 3? Disregarding the importance of making the idea "exactly covers" mathematically precise, observe that the process helps us to define a measure for $\overline{AB}$ in the more general and difficult case when no one of the points P_1, P_2, ... coincides with B. For suppose B falls between two consecutive points of our sequence, say (P_4BP_5). Clearly then we will have to assign to $\overline{AB}$ a measure x such that $4 < x < 5$. In other words we have set up a general process which enables us at least to determine an approximation to the measure of $\overline{AB}$, that is to find lower and upper bounds for it.



We do not complete the discussion but indicate how it proceeds. To fix our ideas, suppose (P_4BP_5). To get a better idea of what the measure of $\overline{AB}$ should be we subdivide P_4P_5 into ten congruent subsegments and proceed as above. Precisely, we set up a subsidiary sequence of points Q_1, ... Q_9 which divide $\overline{P_4P_5}$ into ten congruent subsegments. That is we require

$$\overline{P_4Q_1} \cong \overline{Q_1Q_2} \cong \overline{Q_2Q_3} \cong \ldots \cong \overline{Q_9P_5}$$

and

$$(P_4Q_1Q_2), \quad (Q_1Q_2Q_3), \quad \ldots, \quad (Q_8Q_9P_5).$$

If B were to coincide with one of Q_1, Q_2, ..., Q_9 say $B = Q_6$ we assign to $\overline{AB}$ the measure 4.6. If B falls between two of the Q's say (Q_6BQ_7) we require that x, the measure of $\overline{AB}$, satisfy

$$4.6 < x < 4.7.$$

In the latter case we repeat the process by subdividing $\overline{Q_6Q_7}$ into ten congruent subsegments and proceed as before.

I-8. The definition of linear measure. Clearly we have a complex process (though a refinement of a simple idea) which will assign to segment $\overline{AB}$ a definite decimal, terminating or endless. This decimal we define to be the measure or length of $\overline{AB}$.

I-9. Basic properties of linear measure. We write the measure of $\overline{AB}$ ($\overline{UV}$ still being fixed) as AB. Observe that we really have here a function $\overline{AB} \longrightarrow AB$ which associates with each segment a unique positive real number. What are the basic properties of this "measure" function? They are easily grasped intuitively:

(1) $AB = A'B'$ if and only if $\overline{AB} \cong \overline{A'B'}$ --that is, congruent segments and only congruent segments have equal measures;

(2) If (ABC) then $AB + BC = AC$ --that is, measure is additive in a natural sense;

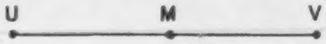
(3) $UV = 1$ --that is, the measure of the unit segment is unity.

Notice that (2) is a clear and useful form of the vague statement, "the whole is the sum of its parts."

We summarize in a theorem which can be deduced from a suitable set of postulates for Euclidean Geometry:

Theorem. Let the segment $\overline{UV}$ be given. Then there exists a function which assigns to each segment $\overline{AB}$ a unique positive real number AB satisfying (1), (2), (3) above.

I-10. Uniqueness of measure function. We naturally ask if there is just one measure function. Clearly not. For the function must depend on the choice of the unit segment $\overline{UV}$. To be specific, suppose we take as a new unit segment, $\overline{UM}$, where M is the mid-point of $\overline{UV}$ that is $\left(\overline{UM} \cong \overline{MV} \text{ and } (UMV)\right)$. Then according to our theorem there will be a new measure function such that $UM = 1$. We see quickly that $UV = 2$; further it can be shown that AB interpreted as the new measure function is twice AB interpreted as the old measure function for any segment $\overline{AB}$. This is a formal statement of the intuitively obvious fact that "halving the unit of measurement doubles the measure." A corresponding result holds in general:



Theorem. Given any two measure functions on the set of all segments one identified as first and the other as second, there is a real number k such that the second function is k times the first function.

In the preceding example we had $k = 2$. Of course k need not be an integer--it can be any positive real number, rational or irrational. As a related example consider the corresponding situation in the measure of angles: The radian measure of an angle is $\frac{\pi}{180}$ times the degree measure of the angle.

Summary: Any two measure functions on the set of all segments are proportional.

What does this mean for the development of the theory of measurement of segments? It says in effect that it doesn't matter which measure function we choose, since making a different choice would only multiply all measures by a constant. Thus, in conventional geometrical theory, we fix a unit $\overline{UV}$ at the beginning, determine a corresponding measure function, and thereafter use this measure function as if it were the only possible one. And instead of saying precisely the measure of $\overline{AB}$ in terms of unit $\overline{UV}$, we say simply the measure of $\overline{AB}$, and forget about $\overline{UV}$. The situation in everyday life is quite different--we employ measure functions based on a variety of units: inches, light years, millimeters, miles.

We close this part of our discussion by observing that the distance between A and B is merely defined to be the measure of $\overline{AB}$. Sometimes we want to refer to the distance between A and A itself, this we take to be zero--a separate definition is required for this case since we may not refer to the segment $\overline{AB}$ unless $A \neq B$.

Query. Was it necessary to use the integer ten in the subdivision process? Would others work? Could the process be simplified by making a different choice?

Part II. Congruence Based on Distance

In this part we discuss the treatment of congruence adopted in the text, contrasting it with the conventional one. The point of departure is to "reverse" the conventional treatment and define congruence in terms of distance. This enables us to use our knowledge of the real number system early in the discussion--it leads to a new treatment of the important geometric relation, betweenness, and a new way of conceiving segments and rays. Since we have seen in the previous discussion that only measure of distance is needed, the following discussion presumes that only one such measure is used.

II-1. *The student's viewpoint*. The conventional treatment, in brief, begins with an undefined notion of congruence of segments and deduces the existence of a distance function from a suitable set of postulates. The high school student--in studying this treatment--somehow absorbs the idea that segments (and angles) can be measured by numbers, and is permitted to apply his knowledge of algebra whenever it is convenient.

II-2. *The Distance Postulate*. Since the student *thinks* of segments and angles as measurable by numbers and it is hopeless to *prove* this at his level from non-numerical postulates, it seems most reasonable to make the existence of a measure function or distance a basic postulate which is used consistently throughout the course. Since we are concerned, because of our previous discussion, with only one unit of measurement, we continue our discussion in terms of the following modification of Postulate 10 of the text.

Postulate 10'. (the Distance Postulate) To every pair of points there corresponds a unique non-negative number. This number is zero if and only if the points of the pair are the same.

If the points are P and Q, then the distance between P and Q is defined to be the non-negative number of Postulate 10', denoted by PQ.

Don't read into this more than it says--it is a very weak statement. Notice that it doesn't state a single property of distance--merely that there is such a thing. In particular it doesn't say anything about lengths of segments--in fact we don't even have segments at this stage of our theory.

II-3. *The Distance Postulate causes a change in viewpoint*. This may seem strange, but it isn't. Most texts begin with a discussion of points and lines in a plane, including such basic ideas as *segment* and *ray*. As in Euclid these ideas essentially are taken as undefined. But having adopted the Distance Postulate we *can* define them. This is an important--and unforeseen--consequence of the Distance Postulate: We don't get just Euclid with the theorems rearranged, but new insights into the basic geometric ideas and a new way of inter-relating them.

II-4. *"Between" and "segment" as defined terms*. How then can we *define* segment in terms of the basic terms point, line, plane? It is easy to do this using the additional notion of a point being *between* two points. Having adopted Postulate 10', the idea of distance is at our disposal and we can define betweenness using Theorem 3-9, The Betweenness-Distance Theorem and related ideas from the text, as follows.

Definition. Let A, B, C be three distinct collinear points. If $AB + BC = AC$ we say B is *between* A and C, and we write (ABC).

We now define segment in terms of betweenness.

Definition. Let A, B be two points. Then *segment* $\overline{AB}$ is the set consisting of A and B together with all points that are between A and B. A and B are called *endpoints* of $\overline{AB}$. Further we define AB to be the *measure* or *length* of $\overline{AB}$.

(This reverses the procedure in the text in which segment was defined first and then betweenness in terms of interior points of a segment. Either way is acceptable.)

Note that the length of a segment is merely the number which is the distance between its endpoints. The contrast with conventional theory is striking. In traditional geometry congruence of segments is basic and a difficult argument is needed to prove the existence of a measure function--here distance is basic and the proof of the existence of a measure function is trivial.

II-5. *Congruence of segments by definition*. Now it is absurdly easy to *define* congruence of segments.

Definition. $\overline{AB} \cong \overline{CD}$ means that the lengths of $\overline{AB}$ and $\overline{CD}$ are equal, that is $AB = CD$.

Formally what we have done is just this. We took the basic property relating congruence and measure ((1) of Section I-9 of this Talk),

$$AB = CD \text{ if and only if } \overline{AB} \cong \overline{CD},$$

which is a theorem in the conventional treatment, and adopted it as a definition in our treatment. There, segments which were congruent were proved to have the same measure--here, segments which have the same measure are called congruent.

II-6. Properties of congruent segments. Does congruence of segments, as we have defined it, have the properties we expect? We see quickly that $=$ is an equivalence relation, that is

(1) $\overline{AB} \cong \overline{AB}$;

(2) If $\overline{AB} \cong \overline{CD}$ then $\overline{CD} \cong \overline{AB}$;

(3) If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$ then $\overline{AB} \cong \overline{EF}$.

These merely say

(1') $AB = AB$;

(2') If $AB = CD$ then $CD = AB$;

(3') If $AB = CD$ and $CD = EF$ then $AB = EF$;

which are the basic properties of equality of numbers.

Further we have

(5) Suppose $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$, (ABC) and $(A'B'C')$. Then $\overline{AC} \cong \overline{A'C'}$.

To prove this we have

$$AB = A'B',$$

$$BC = B'C',$$

so that

$$AB + BC = A'B' + B'C'.$$

The betweenness relations yield

$$AB + BC = AC, \qquad A'B' + B'C' = A'C',$$

and we get

$$AC = A'C' \text{ or } \overline{AC} \cong \overline{A'C'}.$$

Thus several of Euclid's (or Hilbert's) Postulates for congruence reduce, in our treatment, to elementary properties of real numbers.

II-7. The Ruler Postulate. You may wonder if we can also derive from Postulates 1 through 9 and 10' the Location Property: (4), Section I-5:

Let $\overrightarrow{AB}$ be a ray and let $\overline{CD}$ be a segment. Then there exists a unique point P in $\overrightarrow{AB}$ such that $\overline{AP} \cong \overline{CD}$. The answer is--with a vengeance--no. On the basis of Postulates 1 through 9 and 10', we can't prove that a line contains more than two points. Clearly Postulates 1 through 9 and 10' are too weak to support the kind of theoretical structure we are trying to build. For this reason we add to our postulational structure the Ruler Postulate (expressed here in terms of a single unit and blending our definition of coordinate system and our Postulate 12):

Postulate 12'. (The Ruler Postulate.) There is a one-to-one correspondence between the set of all points on a line and the set of real numbers such that:

The distance between two points is the absolute value of the difference of the corresponding numbers.

This powerful postulate guarantees at one swoop that a line has the intrinsic properties we expect of it. Now the lines in every model of our theory will be well-behaved and richly endowed with points. It implies the congruence and order properties of a line in the conventional theory. Specifically it yields: (1) a form of the Point Plotting Theorem (Theorem 3-8); (2) that a segment can be "divided" into a given number of congruent "parts"--in particular it can be bisected (Theorem 3-3). It implies important order properties: Theorem 3-7 which says in effect that the order of points on a line in terms of geometric betweenness corresponds exactly to the order of their coordinates in terms of algebraic betweenness; and the Line Separation Property which is discussed on page 83 and in Section 4-2 of the text.

Observe the attractive inter-dependence of the weak Distance Postulate and the powerful Ruler Postulate. The first asserts the existence of a distance function but permits it to be completely trivial--the second tailors the line to our expectations but is impossible of statement without the notion of distance postulated in the first.

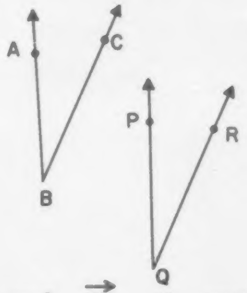
Our discussion suggests an important point in mathematical or deductive thinking. The Distance Postulate enables us to define betweenness but not to prove the existence of a single point between two given points. The Ruler Postulate, however, implies the existence of infinitely many points between any two. This exemplifies the point that a mathematical definition does not assert the existence of the entity defined. You may characterize the pot of gold at the end of the rainbow with great precision but you may experience equally great disappointmetn if you start to search for it before proving an existence theorem.

A final word. We may have oversold the deductive power of the Ruler Postulate and given you the impression that Postulates 1 through 9, 10' and 12' are sufficient for a complete theory of congruence. This is not so. Our theory so far is sufficient for the "linear" theory of congruence, specifically for congruence of segments--but not for congruence of more general figures like angles, triangles, circular arcs or triangular pyramids. For this we must introduce further postulates concerning congruence of angles and triangles. We discuss this in the next part since our main object here has been to indicate the flavor of the treatment in the text in contrast with the conventional one.

Part III. Congruence for Arbitrary Figures and Rigid Motions.

In this part we continue the discussion of congruence by indicating how it is successively defined for familiar elementary figures: angles, triangles, etc. Then using the simple and powerful modern idea of transformation we formulate the congruence concept for arbitrary figures--this surpasses in elegance and generality anything obtained in the field by the classical geometers. As a by-product we obtain--after two millenia--a precise mathematical concept of rigid motion. This was a great cultural achievement of late nineteenth century mathematicians. Rescuing from the jungles of physical intuition Euclid's crude superposition argument, they refined and perfected it to yield an objectively formulated concept which will be of use to human beings as long as they are impelled to think precisely about space.

III-1. Congruence of angles. The conventional treatment of angle congruence is similar to that sketched in Part I for congruence of segments--but naturally it is a bit more complicated since angles are more complex figures than segments. It begins with an undefined relation $\angle ABC \cong \angle PQR$ between two angles which as usual indicates that they are replicas of each other. This may be interpreted concretely to mean that if a frame composed of two jointed rods is set so that the rods coincide with the rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$, then without changing the setting the rods can be made to coincide with $\overrightarrow{QP}$ and $\overrightarrow{QR}$. We assume as for segments that congruence of angles is an equivalence relation:



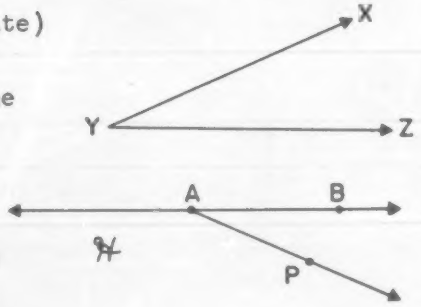
(1) (Reflexive Law) $\angle ABC \cong \angle ABC$;

(2) (Symmetry Law) If $\angle ABC \cong \angle PQR$ then $\angle PQR \cong \angle ABC$;

(3) (Transitive Law) If $\angle ABC \cong \angle PQR$ and $\angle PQR \cong \angle XYZ$ then $\angle ABC \cong \angle XYZ$.

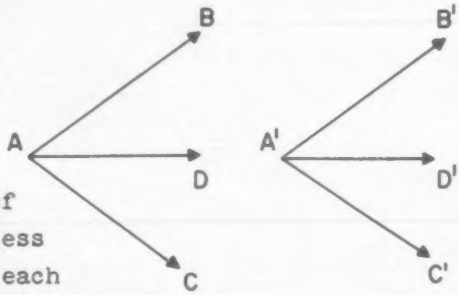
(The Location Postulate for segments (4), Section I-5 of this Talk) has the analogue

(4) (Angle Location Postulate) Let $\angle XYZ$ be any angle and $\overrightarrow{AB}$ be a ray on the edge of half-plane $\mathcal{H}$. Then there is exactly one ray $\overrightarrow{AP}$, with P in $\mathcal{H}$, such that $\angle PAB \cong \angle XYZ$.



And the Additivity Postulate ((5), Section I-5 of this Talk) appears in the form

(5) (Angle-Additivity Postulate) Suppose $\angle BAD \cong \angle B'A'D'$, $\angle DAC \cong \angle D'A'C'$, D is in the interior of $\angle BAC$ and D' is in the interior of $\angle B'A'C'$. Then $\angle BAC \cong \angle B'A'C'$.



Essentially on the basis of these postulates a measure process can be set up which assigns to each angle a unique positive real number called its measure in such a way that a fixed preassigned angle ("unit" angle) has measure 1 (compare Sections I-6 to I-9).

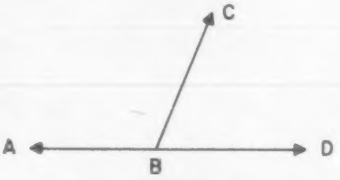
Denoting the measure of $\angle XYZ$ by $m\angle XYZ$, we have as you would expect from our discussion of measure of segments:

(1) $m\angle ABC = m\angle A'B'C'$ if and only if $\angle ABC \cong \angle A'B'C'$;

(2) If C is interior to $\angle ABD$ then $m\angle ABC + m\angle CBD = m\angle ABD$.

(Compare (1), (2) Section I-9 of this Talk).

But there are two properties which are unique to angular measure. First there is a real number b which is a least upper bound for the measure S of all angles (b is 180 in the familiar "degree measure"). Second the measure S of "supplementary adjacent" angles (i.e., a linear pair) always have a constant sum and this sum is the least upper bound b. Stated precisely: If $\angle ABC$ and $\angle CBD$ are a linear pair, then $m\angle ABC + m\angle CBD = b$.



III-2. Congruence of angles based [illegible] angular measure. We saw in (1) above that the conventional theory of angle congruence yields (as for segments) that two angles are congruent if and only if they have equal measures. This suggests (as for segments) that we assume the existence of angular measure and define congruence of angles in terms of it. Thus the treatment in the text assumes

Postulate 16. There exists a correspondence which associates with each angle in space a unique number between 0 and 180.

This number which corresponds, by Postulate 16, to an angle $\angle ABC$ is called the measure of the angle, and is written as $m\angle ABC$.

Clearly our postulate has been set up so that the unit angle is the degree. In other words the angle characterized by $m\angle ABC = 1$ is what is usually defined to be a degree and will have the property that ninety such angles laid "side by side" will form a right angle. Precisely speaking the measure of a right angle will turn out to be 90. Notice that the measure of no angle can be 0 or 180 since our definition of angle restricts the side S to be non-collinear. (For a discussion of this restriction see Commentary for Teachers, Chapter 4.)

Now following a familiar path (Section II-5) we adopt the

Definition. $\angle ABC \cong \angle PQR$ means that $m\angle ABC = m\angle PQR$. Then properties (1), (2), (3) of III-1 above reduce to familiar equality properties of real numbers. A postulate is needed from which (4) and (5) above can be deduced. It may be introduced in terms of ray coordinates.

Definition. Let V be a point in a plane M. A *ray-coordinate system* in M relative to V is a one-to-one correspondence between the set of all rays in M with end-point V and the set of all numbers x such that $0 \leq x < 360$ with the following property: if numbers r and s correspond to rays $\overrightarrow{VR}$ and $\overrightarrow{VS}$ in M and if $r > s$, then

$$m\angle RVS = r - s, \text{ if } r - s < 180;$$

$$m\angle RVS = 360 - (r - s), \text{ if } r - s > 180;$$

$\overrightarrow{VR}$ and $\overrightarrow{VS}$ are opposite rays, if and only if $r - s = 180$.

Definition. The number which a given ray-coordinate system assigns to a ray is called the *ray-coordinate* of the ray. The ray whose ray-coordinate is zero is called the *zero-ray* of the ray-coordinate system.

The Protractor Postulate is then stated as follows:

Postulate 17. (The Protractor Postulate) If M is any plane and if $\overrightarrow{VA}$ and $\overrightarrow{VB}$ are noncollinear rays in M, then there is a unique ray-coordinate system in M relative to V such that $\overrightarrow{VA}$ corresponds to 0 and such that every ray $\overrightarrow{VX}$ with X and B on the same side of $\overrightarrow{VA}$ corresponds to a number less than 180.

III-3. *Congruence of triangles*. We are now ready to consider congruence of triangles. Our definition of congruent triangles (Chapter 5 of text) is essentially the conventional one: One triangle is a "copy" of the other in the sense that its parts are "copies" of the corresponding parts of the other. But observe the precision with which it is formulated. The correspondence doesn't depend on individual interpretation of the vague term "corresponding" but is based objectively on a pairing of the vertices

$$A \longrightarrow A', \quad B \longrightarrow B', \quad C \longrightarrow C'$$

which induces a pairing of sides and of angles

$$\overline{AB} \longrightarrow \overline{A'B'}, \quad \overline{BC} \longrightarrow \overline{B'C'}, \quad \overline{CA} \longrightarrow \overline{C'A'}$$

$$\angle A \longrightarrow \angle A', \quad \angle B \longrightarrow \angle B', \quad \angle C \longrightarrow \angle C'.$$

Notice how spelling out the notion "corresponding" in this way helps to point up the importance of the notion of a congruence which is not mentioned in the conventional treatment. Thus our treatment brings to the fore the idea of a 1-1 correspondence between the vertices of ΔABC and $\Delta A'B'C'$ which ensures that they are congruent because it requires corresponding sides and corresponding angles to be congruent, that is to have equal measures. This simple idea is capable of broad generalization.

Do we need postulates on congruence of triangles? We have a lot of information on congruence of segments and congruence of angles, separately, but nothing to inter-relate these ideas. For example, we can't yet prove that the base angles of an isosceles triangle are congruent. Thus we introduce the S.A.S. Postulate to bind together our knowledge of segment congruence and angle congruence.

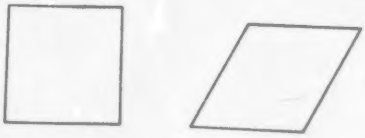
Now let us examine more closely the notion of congruence of triangles. Is it really necessary to require equality of measure of six pairs of corresponding parts? If we think of the sides of a triangle as its basic determining parts it seems very natural to define congruent triangles as having corresponding sides which are congruent. Naturally if we were to adopt this definition we would postulate that if the corresponding sides of two triangles are congruent their corresponding angles also are congruent, in order to ensure that this definition of congruent triangles is equivalent to the familiar one. Notice how much simpler the definition of a congruence between triangles becomes if we adopt the suggested definition. It is merely a 1-1 correspondence between the vertices of the triangles,

$$A \longrightarrow A', \quad B \longrightarrow B', \quad C \longrightarrow C'$$

which "preserves" distances in the sense that the distance between any two vertices of one triangle equals the distance between their corresponding vertices in the second triangle, that is

$$AB = A'B', \qquad BC = B'C', \qquad AC = A'C'.$$

III-4. Congruence of quadrilaterals. The main objection to the suggested definition in the preceding section is that it doesn't generalize in the obvious way for polygons--not even for quadrilaterals.



For example, a square and a rhombus can have sides of the same length and not be congruent. So to guarantee congruence of quadrilaterals it is not sufficient to require just that corresponding sides be congruent, and it is customary to require in addition the congruence of corresponding angles. Thus the conventional definition requiring congruence both of sides and of angles applies equally well to triangles and quadrilaterals.

However angles, though very important, are rather strange creatures compared to segments and it seems desirable, if possible, to characterize congruent quadrilaterals in terms of congruent segments, or equivalently, equal distances. This is not so hard.

Going back to a triangle we observe that its three vertices taken two at a time yield three segments or three distances and that the figure is in a sense determined by these three distances. Similarly the four vertices of a quadrilateral yield not four, but six segments (the sides and the diagonals) and six corresponding distances, which serve to determine the quadrilateral. This suggests: If we have a 1-1 correspondence

$$A \longrightarrow A', \qquad B \longrightarrow B', \qquad C \longrightarrow C', \qquad D \qquad D'$$

between the vertices of the quadrilaterals ABCD, A'B'C'D' such that corresponding distances are preserved, that is

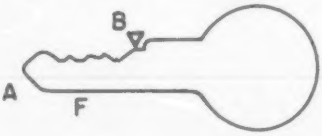
$$AB, AC, AD, BC, BD, CD = A'B', A'C', A'D', B'C', B'D', C'D',$$

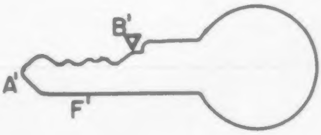
respectively, we call the correspondence a *congruence* and we write ABCD = A'B'C'D'. It is not hard to show that this definition is equivalent to the more familiar one.

III-5. *Congruence of arbitrary figures*. We now must face the problem of formulating a general definition of congruence. The piecemeal process we have employed, defining congruence separately for segments, angles, triangles, quadrilaterals is unavoidable in an elementary treatment but is neither satisfying nor complete. For it still remains to define congruent circles and congruent circular arcs and congruent ellipses and congruent rectangular solids, etc. In each case we construct an appropriate definition, we are sure it is correct, and are equally sure that the general concept has eluded us.

So let's make a fresh start. Suppose F and F' are two congruent figures. Our basic intuition is that F' is an exact copy of F. Somehow this entails that each "part" of F' copies a corresponding "part" of F--that each point of F' behaves like some corresponding point of F. If F has a sharp point at A then F' must have a sharp point at a corresponding point A'; if F has maximum flatness at B then F' has maximum flatness at a corresponding point B'; if F has a largest chord $\overline{PQ}$ of length 12.3 then F' has a corresponding largest chord $\overline{P'Q'}$ of the same length, 12.3; and so on. How can we tie together these illustrations in a simple and precise way?

III-6. *A congruence machine*. Suppose instead of conceiving F' as a given copy of F, we take F and try to make a copy F' of it. As an illustration let F be a house key. Then F' can be produced by a key duplicating machine. The machine has the secret of the congruence concept--how does it work?





The machine has two moving parts: a scanning bar which traces the given key and a cutting bar which cuts a blank into a duplicate. As the scanning bar traces F starting at its tip A, the cutting bar traces the blank starting at its corresponding tip A'. As the scanner moves to position B, the cutter cuts away the metal and comes to rest at a corresponding position B'. When B rises to a "peak" so does B'--when B falls to a trough so does B'--when B traverses a line segment, B' traverses a line segment of equal length.

What guarantees that this process yields a true copy? Simply this: When the scanner is fixed in a position B, the cutter comes to rest in a position B' such that distances AB and A'B' are equal. And this is true for each position B of the scanner. Clearly what the machine does is to associate to each chord $\overline{AB}$ from A of F an "equal" chord $\overline{A'B'}$ from A' of F'. And it associates the chords by associating their endpoints B and B'. Precisely speaking, the machine effects a 1-1 correspondence $X \longleftrightarrow X'$ between F and F' such that the distance AX always equals the distance A'X'.

Does this property hold just for A, the tip of F, and A' its correspondent in F'? Clearly not. The machine doesn't know where we start. What we have asserted about the chords of F from A will hold just as well for the chords from any point of F. So the 1-1 correspondence $X \longleftrightarrow X'$ between F and F' has the stronger property that for every choice of P and Q if $P \longleftrightarrow P'$, $Q \longleftrightarrow Q'$ then $PQ = P'Q'$, or as we say the correspondence preserves distance. Here we have the essence of the concept of congruence.

The legend has it that when Pythagoras succeeded in proving the theorem ascribed to him, he was so elated that he sacrificed the hecatomb of oxen to the gods. Surely in the light of this tradition the formal definition of congruence deserves a section all to itself.

III-7. The definition. Let $X \longleftrightarrow X'$ be a 1-1 correspondence between two sets of points F, F' such that

$$P \longleftrightarrow P', \qquad Q \longleftrightarrow Q'$$

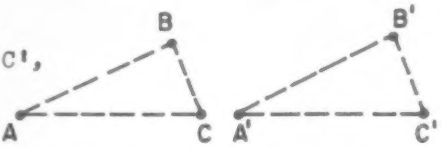
always implies $PQ = P'Q'$. Then we say F is congruent to F' and we write $F = F'$. Moreover we call the 1-1 correspondence a congruence between F and F'.

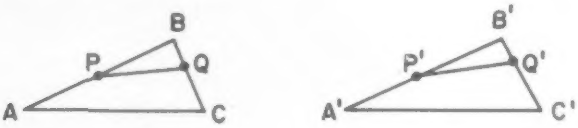
This definition is the culmination of two thousand years of thinking about congruence. Although it may seem quite abstract it unifies and unites the piecemeal discussion of congruence we have given. Every instance of congruent figures discussed above from segments to quadrilaterals can be proved to be a case of our general definition. This is discussed in detail in the Appendix on Rigid Motion in the text.

As a simple illustration of the definition let F and F' be triples of noncollinear points, say F is $\{A, B, C\}$ and F' is $\{A', B', C'\}$. Let the 1-1 correspondence between F and F' which preserves distance be

(1) $A \longleftrightarrow A'$, $B \longleftrightarrow B'$, $C \longleftrightarrow C'$.

Then we have $AB = A'B'$, $BC = B'C'$, $AC = A'C'$. We see intuitively that F' is a copy of F. Now shift from the point triples to the triangles they determine. The S.S.S. Theorem tells that ΔABC is congruent to $\Delta A'B'C'$ in the conventional sense.

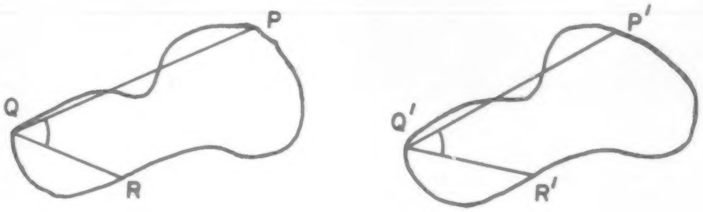




It follows (see the Appendix on Rigid Motion) that $\Delta ABC \cong \Delta A'B'C'$ in the sense of our definition. Actually there is a 1-1 correspondence between the infinite point sets ΔABC, $\Delta A'B'C'$ which makes the vertices correspond as in (1) and which has the property that $P \longleftrightarrow P'$, $Q \longleftrightarrow Q'$ always implies $PQ = P'Q'$.

Observe how the correspondence between the triangle is engendered by the trivial seeming correspondence between their vertices. For example, if P is on $\overline{AB}$ its correspondent P' is determined as the unique point P' on $\overline{A'B'}$ such that $A'P' = AP$. Let us think of the finite set of its vertices, {A, B, C}, as a "skeleton" of ΔABC. Then if the skeletons {A, B, C}, {A', B', C'} of two triangles are congruent the triangles as a whole are congruent--using "congruent" in its present sense. This idea was too complex to introduce in Chapter 5 of the text. But it was fore-shadowed there in the insistence that congruence of triangles was the consequence of the existence of a "congruence" between them--that is, a 1-1 correspondence between their sets of vertices which preserves lengths of sides and measures of angles.

There is an essential element of complexity in the definition of congruence: It requires (in general) the pairing off of the points of two infinite sets so as to preserve distance. This is unavoidable--it even seems to be present in the comparatively simple problem of duplicating keys. There is however an important element of simplicity: We don't have to mention angles and the preservation of their measures--the distance concept covers the situation. It follows easily that angle measures are preserved:



for if $P \longleftrightarrow P'$, $Q \longleftrightarrow Q'$, $R \longleftrightarrow R'$ correspond under a congruence between F and F', and P, Q, R are non-collinear, we see by the S.S.S. Theorem that $m\angle PQR = m\angle P'Q'R'$.

You may find it interesting to give for quadrilaterals a discussion like the above for triangles--consider the vertex sets {A, B, C, D}, {A', B', C', D'} of quadrilaterals ABCD, A'B'C'D' as their "skeletons." In this connection recall the discussion of congruence of quadrilaterals at the end of Section III-4.

III-8. Motion in geometry. We can state the definition of rigid motion now, but it probably will be more meaningful if we say a few words first about the sense in which "motion" is used in contemporary geometry.

Let a body B move physically from an initial position F in space to a final position F'. It is not necessary for our purposes in geometry (as compared say with kinematics or fluid dynamics) to bother about the intermediate stages of the motion. So we can describe the motion merely by specifying the initial position X in F of an arbitrary point P of body B and its corresponding final position X' in F'. In its most general form, then, a motion is conceived as a 1-1 correspondence or transformation between two figures F and F'. The technical term "transformation" is often preferable to "motion" since it doesn't suggest various irrelevant attributes of physical motion.

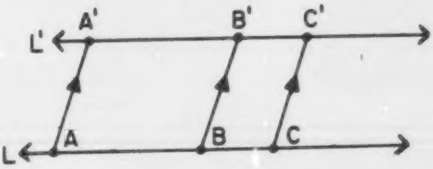
III-9. Rigid Motion. A motion or transformation between two point sets F and F' is a rigid motion if it preserves distances--that is if it is a congruence between F and F' as defined in Section III-7. A detailed discussion of the concept of rigid motion appears in an Appendix to the text.

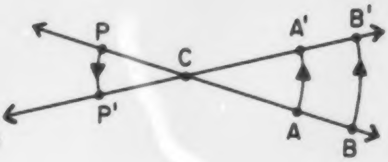
To introduce you to the modern theory of congruent figures and rigid motion we have put the main emphasis on the first, since it is more familiar and seems easier to apprehend. However, glancing back at the definition of congruent figures, you see that it implicitly involves the notion of rigid motion. In fact now we can reword it: F is congruent to F' provided there exists a rigid motion between them, or as we say more graphically, a rigid motion which "transforms F into F'". This is the highly refined culmination of the vague and famous classical statement which served to introduce our discussion of congruence: "Two figures are congruent if they can be made to coincide by a rigid motion."

Sometimes the clarification of the basic concepts of a branch of mathematics firms up the foundations, puts the capstone on the superstructure and sets it to rest. This is not so here. The concept of rigid motion has stimulated the study of classical geometry, has yielded new insights and helped to unfold new unities. It has suggested the study of more general geometric transformations ("non-rigid motions") and has presented problems to the field of Modern Algebra, since motions tend to occur in certain "natural algebraic formations" called groups.

In the first place congruence and rigid motion have an impact on geometry since they apply to all figures. We can talk precisely not merely about congruence of (or rigid motion between) triangular pyramids or spherical zones or hyperbolic paraboloids but also of lines, planes, space, half-planes, rays, etc. At first it may sound silly to say a line is congruent to a line--but try to find a better replica of a line than a line! It must be just because the relation congruence applied to lines is so fundamental and universal that we are not conscious of it--as a fish must be unconscious of the notion humidity. In a first approach, congruence takes on importance as applied to segments (or angles or triangles) precisely because not all segments (or angles or triangles) are congruent to each other.

So it may seem trivial to say a line is congruent to a line or a plane to a plane or space to itself. But suppose we shift the focus from the static idea of congruent figures to the dynamic--and logically prior--idea of rigid motion. Is it trivial to say there exist rigid motions between lines or between planes or between space and itself? Just to ask this question discloses a broad vista: One of the principal concerns of contemporary geometry (or contemporary mathematics) is the study of transformations (rigid and non-rigid) of n-dimensional spaces.

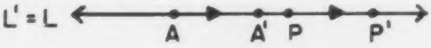


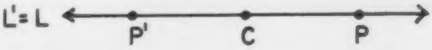


Consider the simplest case: Rigid motions which transform a line L into a line L'. If L L' we have slides or translations which "move" the points of L along parallel transversals to get their corresponding points of L'. If L and L' meet in just one point C we have a rotation about C. If L and L' coincide, that is $L = L'$, we have two types of rigid motions operating on L:

(1) translations along L;

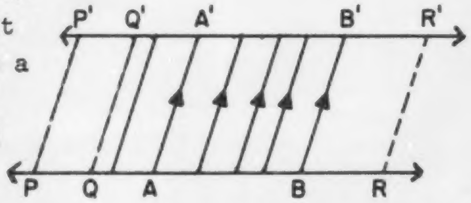
(2) reflections of L in a point C, where point C of L is "fixed" (that is it corresponds to itself) and every other point of L "moves" on L from one side of C to the other.





Similar considerations apply to planes. The theory culminates in the study of rigid motions of space--that is between space and itself. Here the basic types are translations, in which no point is fixed, rotations in which each point of a line (the axis of the rotation) is fixed, and reflections in a plane $\mathcal{E}$ in which each point of plane $\mathcal{E}$ is fixed and the half-spaces separated by $\mathcal{E}$ are "interchanged." More precisely a reflection in $\mathcal{E}$ is a transformation $X \longleftrightarrow X'$ such that if X is in $\mathcal{E}$ then $X' = X$ and if X is not in $\mathcal{E}$ then $\mathcal{E}$ is the perpendicular bisector of $\overline{XX'}$. All rigid motions of space are "combinations" of these three basic types, just as all positive integers other than 1 are combinations of primes.

You may say that the theory of rigid motions of lines, planes and space is attractive and relatively simple, but haven't we left out the annoying complexities involved in the study of specific congruent figures like segments, truncated triangular pyramids and cones with oval bases? Not at all! They are elegantly covered in the theory of rigid motions of the basic "linear manifolds": line, plane, space.



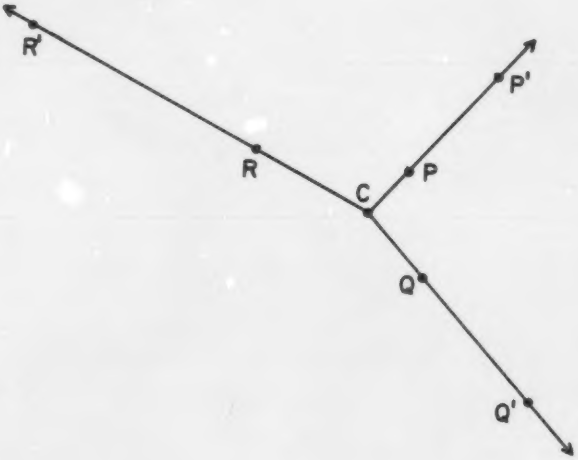
As a very simple illustration suppose segment $\overline{AB}$ is congruent to segment $\overline{A'B'}$. Then there is a rigid motion between them which, let us say, makes A correspond to A' and B to B'. Now we have the remarkable result that this rigid motion, which is a certain kind of 1-1 correspondence between segments $\overline{AB}$ and $\overline{A'B'}$ can be extended to form a rigid motion between the whole line $\overleftrightarrow{AB}$ and the whole line $\overleftrightarrow{A'B'}$--and this extension can be made in just one way. Thus we don't disturb the correspondence between $\overline{AB}$ and $\overline{A'B'}$ but "amplify" it by suitably defining a unique correspondent for each point of $\overleftrightarrow{AB}$ not in $\overline{AB}$, so that the final correspondence is a rigid motion between $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$. So in the study of rigid motions between lines as wholes, we are automatically covering all possible rigid motions (and hence all possible relations of congruence) between "linear" figures; (that is, subsets of lines which contain more than one point). Similarly any rigid motion between "planar" figures (that is, subsets of a plane which are not contained in any line) is uniquely extendable to a rigid motion of their containing planes. Finally we observe that any conceivable rigid motion is encompassed by a rigid motion of space.

III-10. Non-rigid motions. As we have indicated, modern geometry is concerned with transformations that do not preserve distance as well as with those which do. In Euclidean Geometry the most important example is a similarity, which bears the same relation to similar figures that a congruence or rigid motion does to figures which are congruent. Formally suppose

$X \quad X'$ is a 1-1 correspondence between figures F and F' such that

$$P \longleftrightarrow P', \qquad Q \longleftrightarrow Q'$$

always implies $P'Q' = k \cdot PQ$ where k is a fixed positive number. Then we call the correspondence a similarity transformation or a similarity and we say F is similar to F'. It easily follows that a similarity transformation--although it is not in general a rigid motion--always preserves angle measures. This definition of similar figures, when restricted to triangles, can be proved equivalent to the familiar one. The simplest general type of similarity is the dilatation (in a plane or in space)--this is a similarity which leaves a given point C fixed and radially "stretches" the distance of any point from C by a positive factor k.



Other important types of transformations are central in various geometric theories. For example, "parallel projection" between planes in affine geometry; "central projection" between planes in projective geometry; and topological transformations, which are a type of continuous 1-1 correspondence, in topology. The theory of map-making is concerned with various "projections" or other kinds of transformations between a sphere and a cone, cylinder or plane.

And so we have ended our talk by touching upon a modern generalization of rigid motion which well might merit a talk for itself.

SOME REMARKS ON STUDIES II

The Studies in Mathematics, Volume II, Euclidean Geometry Based on Ruler and Protractor Axioms was prepared to help teachers become familiar with the approach to Euclidean Geometry which has been adopted by SMSG. It was intended for use in summer courses for teachers, and is not suitable either as a text for high school students or as a teachers' manual. This volume of Studies probes into the foundations of geometry, covering material which is too difficult for most high school students, but which it is desirable for teachers to know. It analyzes the changes in attitude toward geometrical ideas,from Greek times to the present, and traces the modifications of geometrical terminology, notation, and other aspects of expression which the changes in attitude have brought about.

In particular, Studies II goes into considerable detail about the Ruler and Proptractor Postulates and their implications. Because of the central role played by these postulates in Geometry with Coordinates, Studies II is of special value to teachers of this text.

The remainder of this Talk is devoted to a brief analysis of the differences between the postulate system of Geometry with Coordinates and the form of the postulates discussed in Studies II, so as to make the latter book as useful as possible to teachers of G.W.

(1) The postulates of connection stated in Chapter 3 of Studies II, are called the incidence postulates in G.W. The obvious correspondence between the two sets of postulates is shown in the following table.

	G.W.	Studies II
Postulate	2	1a
	3	2
	5	1b
	6	3
	8	4
	9	5

Postulates 1, 4, and 7 of *G.W.* combined with the other *G.W.* incidence postulates establish Theorem 2-7 which corresponds to Postulate 1c of *Studies II*. In *G.W.*, the statements that lines and planes are sets of points are made in the postulates. In *Studies II*, these statements are made in the remarks preceding the postulates. The mathematical content of these two systems of postulates is exactly the same though they do differ in wording and numbering.

(2) The discussion of distance and the Ruler Postulate in Chapter 4 of *Studies II*, does not involve a consideration of different units of measurement, as treated in *G.W.* The *Talk* on the *Concept of Congruence* makes clear that betweenness and related matters can be adequately handled without changing the units of measurement. If *G.W.* were to express all distances in terms of one unit-pair, then Postulates 11 and 13 would be unnecessary and Postulate 10 would be modified. This modified postulate would state simply that associated with every pair of points is a number called the distance between them, and that this number is positive if the points of the pair are distinct and zero if the points are the same. If the distance between P and Q were denoted by PQ, then PQ = QP just as in the *G.W.* text since there is no preferred order for the points. (In the notation of *Studies II* this equality is written d(P, Q) = d(Q, P).

(3) It is clear that Postulate 12, the Ruler Postulate in *G.W.*, combined with the definition of a coordinate system is equivalent to Postulate 7 of *Studies II*.

(4) The definition of betweenness in *Studies II* is related to the Betweenness - Distance Theorem, Theorem 3-9 of *G.W.* In the discussion of the latter in the text it is pointed out that the distance relation involved in the theorem characterizes betweenness. Thus the equivalence of the two different definitions may be established.

(5) The difference between the Protractor Axioms in *G.W.* and in *Studies II*, lies in the fact that *G.W.* uses a "circular protractor" while *Studies II* uses in the background a "semicircular protractor." As a consequence of using the "circular protractor" there is a close analogy between the Protractor Postulate and the Ruler Postulate in *G.W.* since both deal with coordinate systems.

This analogy is nowhere near as striking when the "semicircular protractor" is used.

The ray-coordinate system in *G.W.* also leads more naturally to the extension of the concept of angles as needed in trigonometry and to the introduction of polar coordinates in later courses.

We note that Postulate 9 of *Studies II* corresponds to Postulate 16 of *G.W.* Postulate 10 of *Studies II* is similar to Postulate 17 of *G.W.* Postulates 11 and 12 of *Studies II* are consequences of Postulates 16, 17, and 18 of *G.W.* The discussion in *Studies II* of these postulates about angles is concerned primarily with synthetic techniques of proving that certain lines intersect, and kindred matters.

One of the important aspects of coordinate geometry is that, as soon as we are in a position to pursue geometric theory using coordinates, the difficulty of answering many of the questions of synthetic geometry raised in *Studies II* concerning betweenness and the separation postulates disappears. This is because the coordinates of points and equations of lines, planes, circles, spheres, etc. carry with them complete information for settling these questions in a straight-forward algebraic manner.

A careful reading of *Studies II* is bound to be both pleasant and profitable.

A DEVELOPMENT IN WHICH SIMILARITY PRECEDES CONGRUENCE

In the text we first developed the notion of congruent triangles, introduced the parallel postulate, developed the notion of similarity, and then proved the Pythagorean Theorem and its converse. There are other ways of organizing this part of Euclidean geometry. The particular way to be outlined here introduces a similarity postulate first. The notion of congruence then appears as a natural and special instance of similarity. The "parallel postulate" occurs as a theorem, and the Pythagorean Theorem and its converse are easily obtained.

We did not choose to follow this organization in the text for two reasons:

1. It represents a considerable departure from the traditional organization of geometry.

2. It places the discussion of the relation between non-Euclidean and Euclidean geometry into somewhat unfamiliar territory. For instance, the distinction between Hyperbolic and Euclidean geometry hinges in this setting (since the "parallel postulate" is now a theorem in our development of Euclidean geometry) on the fact that in Hyperbolic geometry if the lengths of the sides of one triangle are double those of a second triangle, the two triangles are not similar. The discovery of non-Euclidean geometry is also difficult to explicate in this setting, since, historically, the discovery came about from close study of the role of the parallel postulate. On the other hand, we consider this organization to be attractive because of its conciseness and the consequent rapidity with which we can get to analytic geometry in the plane. We also feel there are many pedagogical advantages to be gained from viewing congruence as a special case of similarity.

In writing an outline of the development based on similarity we will resort to a "telegraphic style": postulate, definition, theorem, proof, etc., with few intervening remarks. If you wish to teach from this outline, you will have to "clothe its bare bones" and supply your own problems. There is material in the text that could easily be adapted with only slight modification. Another source is Basic Geometry by G. D. Birkhoff and R. Beatley, Third Edition, Chelsea Publishing Company, New York, which is organized on a similar plan.

The following outline is an interesting example of the changes in organization that result from choosing a different postulate, even though the postulates as a whole describe the same geometry. We also remark that because of the need for some theorems about parallels in developing analytic geometry we inserted them in the outline in spite of the fact that the Pythagorean Theorem and its converse could have been established immediately following the theorems on perpendiculars. In other words, the Pythagorean Theorem and its converse were not our only goals, since we wished to complete the synthetic framework needed to introduce and discuss plane analytic geometry.

Outline of a Chapter on Similarity and Congruence

1. Introduction

Relationship between floor plans of a building and the floors of the actual building.
Scale drawing.
"Same shape."

2. Proportionality

Definition. The numbers $p, q, r, \dots$, are said to be proportional to the numbers $a, b, c, \dots$, if and only if there is a non-zero number k such that $p = ka$, $q = kb$, $r = kc$, ...

The number k is called the constant of proportionality or proportionality factor.

Examples.

Properties of proportionality.

3. Similarity and Congruence

One-to-one correspondences between the vertices of triangles and notation. (Same as in text omitting any reference to the idea of congruence and replacing such references by appeals to idea of same shape.)

Definition. The six parts of a triangle are its angles and its sides.

Definition. Given a one-to-one correspondence between the vertices of triangles ABC and DEF, $ABC \longleftrightarrow DEF$, the six pair of corresponding parts are: $\angle A$ and $\angle D$, $\angle B$ and $\angle E$, $\angle C$ and $\angle F$, $\overline{AB}$ and $\overline{DE}$, $\overline{AC}$ and $\overline{DF}$, $\overline{BC}$ and $\overline{EF}$.

Definitions. A one-to-one correspondence between vertices of triangles is said to be a similarity if and only if corresponding angles are congruent and the lengths of corresponding sides are proportional.

Triangles are said to be similar if and only if there is a one-to-one correspondence between their vertices that is a similarity. Notation " $\sim$ " .

Theorem. If T_1, T_2, T_3 are triangles, then

(1) T_1 is similar to T_1.

(2) If T_1 is similar to T_2, then T_2 is similar to T_1.

(3) If T_1 is similar to T_2, and T_2 is similar to T_3, then T_1 is similar to T_3.

Proof: Properties of one-to-one correspondences and definition of similarity.

Experiment. To show reasonableness of following postulate.

Postulate. (S.A.S. Similarity Postulate.)

If there is a one-to-one correspondence between the vertices of triangles such that the lengths of two sides of one are proportional to the corresponding lengths of the sides of the other and the angle included by the two sides is congruent to the corresponding angle of the other, then the correspondence is a similarity.

Definitions. A similarity between triangles is said to be a congruence if and only if the proportionality factor between the lengths of the corresponding sides is one.

In this case the triangles are said to be congruent.

Notation. $\Delta ABC \cong \Delta DEF$.

Theorem. If T_1 , T_2 , T_3 are triangles, then

(1) T_1 is congruent to T_1 .

(2) If T_1 is congruent to T_2 , then T_2 is congruent to T_1 .

(3) If T_1 is congruent to T_2 and T_2 is congruent to T_3 , then T_1 is congruent to T_3

Proof: Properties of one-to-one correspondences and definition of congruence.

Theorem. (S.A.S. Congruence Theorem.)

If there is a one-to-one correspondence between the vertices of triangles, such that two sides and the included angle of one are congruent, respectively, to the corresponding sides and angle of the other, then the correspondence is a congruence and the triangles are congruent.

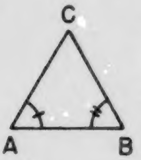
Proof: An immediate consequence of the definition of congruence and the S.A.S. Similarity Postulate.

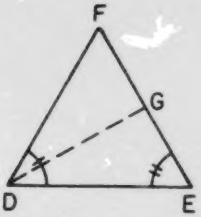
Experiment. To motivate next theorem.

Theorem. (A.A. Similarity Theorem.)

If there is a one-to-one correspondence between the vertices of two triangles such that two angles of one are congruent to the corresponding angles of the other, then the correspondence is a similarity and the triangles are similar.

Proof: Let ABC and DEF be the given triangles, $ABC \leftrightarrow DEF$ the given one-to-one correspondence and $\angle A \cong \angle D$, $\angle B \cong \angle E$.





We have to prove $\Delta ABC \sim \Delta DEF$. Since $AB \neq 0$, there exists $k \neq 0$ such that $DE = kAB$. On $\overleftrightarrow{EF}$ locate G such that $EG = kBC$. Then since $BC \neq 0$, D, E, G are vertices of a triangle and by the S.A.S. Similarity Postulate $\Delta ABC \sim \Delta DEG$. Consequently, $\angle A \cong \angle GDE$. But $\angle A \cong \angle FDE$; therefore, G lies on both $\overleftrightarrow{DF}$ and $\overleftrightarrow{EF}$. Hence G is F . Hence $\Delta ABC \sim \Delta DEF$.

Remark. Note resemblance between this proof and the proof of the A.S.A. Congruence Theorem in Appendix V.

Theorem. (S.A.A. Congruence Theorem.)

If there is a one-to-one correspondence between the vertices of two triangles such that two angles and a side of one are congruent, respectively, to the corresponding angles and side of the other, then the correspondence is a congruence and the triangles are congruent.

Proof: Immediate consequence of A.S.A. Similarity Theorem and definition of congruence.

Definition. A triangle is said to be isosceles if and only if at least two of its sides are congruent.

Definition. A triangle is said to be equilateral if and only if all three sides are congruent.

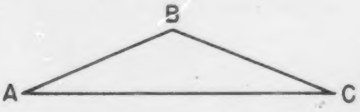
Definitions. In a non-equilateral isosceles triangle the side that is not congruent to either of the other two is said to be the base of the triangle.

The angles of the triangle that contain the base are said to be the base angles of the triangle.

Remark. Any side of an equilateral triangle may be called a base. Once a base has been chosen, then the previous definition can be used to identify base angles.

Theorem. (Base angles of an isosceles triangle are congruent.) If two sides of a triangle are congruent, then the angles opposite them are congruent.

Proof: In ΔABC let $\overline{AB} \cong \overline{BC}$.



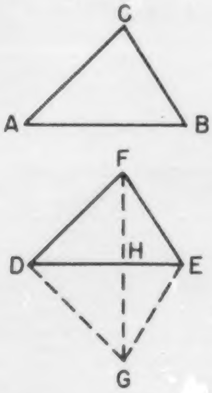
We have to prove $\angle A \cong \angle C$. By S.A.S. Similarity Postulate $\Delta ABC \sim \Delta CBA$. Hence, $\angle A \cong \angle C$.

Theorem. (S.S.S. Similarity Theorem.)

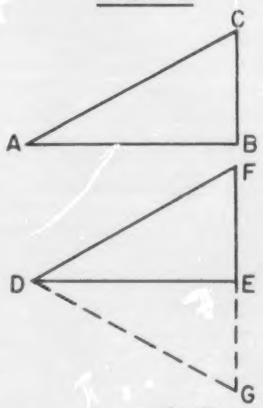
If there is a one-to-one correspondence between vertices of triangles such that corresponding sides are proportional, then the correspondence is a similarity and the triangles are similar.

Proof: Let ABC and DEF be the given triangles, $ABC \longleftrightarrow DEF$ the given one-to-one correspondence. Suppose that DE , EF , FD are proportional to AB , BC , CA with proportionality factor $k \neq 0$. We are to prove $\Delta ABC \sim \Delta DEF$. Due to the construction used in the proof we distinguish three cases.

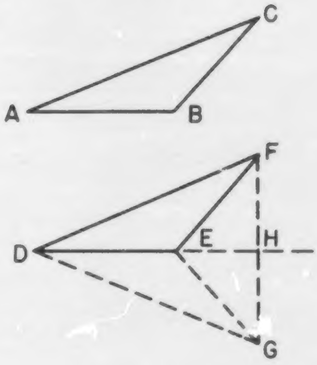
CASE 1



CASE 2



CASE 3



On the opposite side of $\overleftrightarrow{DE}$ from F, there is a point G such that $\angle DEG \cong \angle ABC$ and $EG = kBC$ where k is the proportionality factor. Since $DE = kAB$, $\triangle ABC \sim \triangle DEG$ by the S.A.S. Similarity Postulate. From this similarity we conclude two key facts.

(a) $DG = DF$. (For the similarity implies $DG = kAC$ and by hypothesis $DF = kAC$.)

(b) $EG = EF$. (For the similarity implies $EG = kBC$ and by hypothesis $EF = kBC$.)

For the remainder of the proof, we distinguish three cases according as the point of intersection H of $\overline{FG}$ with $\overleftrightarrow{DE}$ lies between E and D, is an endpoint of $\overline{ED}$, does not lie in $\overline{ED}$. Examples of these cases are illustrated in the diagrams above.

Case 1. (H between E and D.) Since $EG = EF$, $\triangle GEF$ is isosceles. Consequently $\angle EFG \cong \angle EGF$. Furthermore, since $DG = DF$, $\triangle DFG$ is isosceles and $\angle DFG \cong \angle DGF$. Hence $m\angle DFG + m\angle EFG = m\angle DGF + m\angle EGF$. Now $\angle DFG$ and $\angle EFG$ are adjacent angles and $\angle DGF$ and $\angle EGF$ are adjacent angles, so the sums of their measures are, respectively, $m\angle DFE$ and $m\angle DGE$. But we proved the sums of these measures are equal; hence $\angle DFE \cong \angle DGE$. Consequently $\triangle DEF \sim \triangle DEG$ by S.A.S. Similarity Postulate. Since $\triangle ABC \sim \triangle DEG$, this means $\triangle ABC \sim \triangle DEF$.

Case 2. (H an endpoint of $\overline{ED}$.) If $H = E$, then it follows from $DG = DF$ that $\triangle DFG$ is isosceles and $\angle DFE \cong \angle DGE$. Hence $\triangle DEF \sim \triangle DEG$ by the S.A.S. Similarity Postulate. Consequently $\triangle ABC \sim \triangle DEF$. If $H = D$, then it follows from $EG = EF$ that $\triangle EFG$ is isosceles and $\angle DFE \cong \angle DGE$. Consequently, as above, $\triangle ABC \sim \triangle DEF$.

Case 3. (H not in $\overline{ED}$.) In this case, the argument is similar to that of Case 1 except that near the end of the argument a difference of angle measures is involved instead of a sum.

Remark. Note resemblance between this proof and the proof of the S.S.S. Congruence Theorem in Appendix V.

Theorem. (S.S.S. Congruence Theorem.)

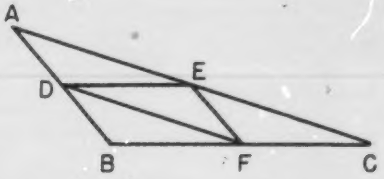
If there is a one-to-one correspondence between the vertices of triangles such that corresponding sides are congruent then the correspondence is a congruence.

Proof: Immediate consequence of S.S.S. Similarity Theorem and definition of congruence.

4. Sum of the Measures of the Angles of a Triangle.

Objective. To show that the sum of the measures of the angles of a triangle is 180.

Experiment. Make a cardboard triangle ABC. Locate the midpoints D, E, F of each of its sides. Label as in the figure.



Cut along the segments joining the midpoints to form four triangles, ADE, DEF, DBF, FEC. Verify that the four triangles seem to be congruent. Are they similar to $\triangle ABC$? Reassemble these triangles to form the original figure and label so that, when the figure is disassembled, you can identify which vertices and sides are corresponding. Do you see that $\angle BFD \cong \angle FCE$, $\angle DFE \cong \angle DAE$, $\angle EFC \cong \angle DBF$? Does this together with the fact that $m\angle BFD + m\angle DFE + m\angle EFC = 180$ suggest a method of proving that the sum of the measures of the angles of $\triangle ABC$ is 180 ?

Theorem. In $\triangle ABC$, if D, E are the midpoints of $\overline{AB}$, $\overline{AC}$, respectively, then $\triangle ABC \sim \triangle ADE$.

Proof: Using the notation of the statement of the theorem, we know that $AB = 2AD$, $AC = 2AE$, $\angle DAE \cong \angle BAC$. Hence by S.A.S. Similarity Postulate $\triangle ABC \sim \triangle ADE$.

Theorem. If D, E, F are the midpoints of the sides $\overline{AB}$, $\overline{AC}$, $\overline{BC}$, respectively, of $\triangle ABC$, then

$$\begin{aligned}
&\triangle ABC \sim \triangle ADE\,,\\
&\triangle ABC \sim \triangle DBF\,,\\
&\triangle ABC \sim \triangle EFC\,,\\
&\triangle ABC \sim \triangle FED\,.
\end{aligned}$$

Proof: Using the notation of the statement of the theorem, we see that the first three items in the conclusion follow from the preceding theorem. From these three items and the fact that D, E, F are midpoints of the sides, we conclude that $BC = 2DE$, $AC = 2DF$, $AB = 2EF$. Hence by S.S.S. Congruence Theorem, $\triangle ABC \sim \triangle FED$, the final item in the conclusion.

Theorem. (Angle Sum Theorem.)

The sum of the measures of the angles of any triangle is 180 .

Proof: Let the vertices of the triangle considered be denoted by A, B, C. Let D, E, F be the midpoints, respectively, of $\overline{AB}$, $\overline{AC}$, $\overline{BC}$. From the previous theorem, $\angle A \cong \angle DFE$, $\angle B \cong \angle EFC$, $\angle C \cong \angle DFB$. Hence $m\angle A + m\angle B + m\angle C = m\angle DFE + m\angle EFC + m\angle DFB$. But $m\angle DFE + m\angle EFC = m\angle DFC$, since $\overrightarrow{FE}$ is between $\overrightarrow{FD}$ and $\overrightarrow{FC}$. Now $m\angle DFE + m\angle DFC = 180$, since these angles form a linear pair. Hence

$$m\angle A + m\angle B + m\angle C = 180\,.$$

Definition. Given a triangle, then an angle contained in the union of the lines containing the sides of the triangle is said to be an *exterior angle* of the triangle if and only if it forms a linear pair with an angle of the triangle.

Remark. Exterior angles of a triangle at the same vertex are congruent.

Theorem. (Exterior Angle Theorem.)

The measure of an exterior angle of a triangle is equal to the sum of the measures of the opposite angles of the triangle.

Proof: Immediate consequence of Definition and Angle Sum Theorem.

Corollary. (Weak form of Exterior Angle Theorem.)

The measure of an exterior angle of a triangle is greater than the measure of either opposite angle of the triangle.

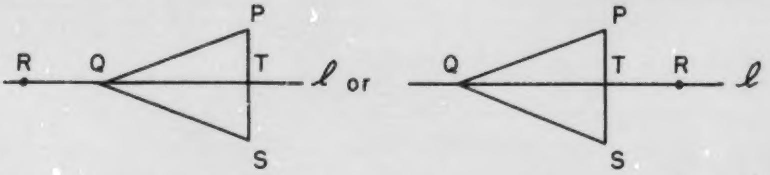
5. Perpendicularity and Parallelism

Theorem. If ℓ is a line and P is a point not on ℓ, then there is at most one perpendicular to ℓ that contains P .

Proof: Use the notation of the theorem and suppose that there exist two distinct perpendiculars to ℓ through P . Let the intersection of these perpendiculars with ℓ be the points Q and R . Then ΔPQR has two right angles. This contradicts the angle sum theorem. Hence there is at most one perpendicular to ℓ containing P .

Theorem. If ℓ is a line and P is a point not on ℓ, then there is at least one perpendicular to ℓ that contains P .

Proof: Use the notation of the theorem. By the previous theorem and the fact that ℓ contains more than one point, there is a point Q on ℓ such that $\overleftrightarrow{PQ}$ is not perpendicular to ℓ. Let R be a point on ℓ not the same as Q. On the side of ℓ that does not contain P, there is a point S such that $\angle PQR \cong \angle SQR$.



The segment $\overline{PS}$ meets ℓ in a point T. ($T \neq Q$, but $T = R$ is possible.) Now $\Delta PQT \cong \Delta SQT$ by S.A.S. Congruence Theorem. Hence $\angle QTP \cong \angle QTS$, but since these angles form a linear pair, $m\,\angle QTP = 90$. Therefore $\overleftrightarrow{PS}$ is perpendicular to ℓ.

Remark. These two theorems can be combined with the result obtained in Chapter 4 to give the following theorem.

Theorem. If in a plane, ℓ and P are respectively a line and a point then there is exactly one line in the plane that is perpendicular to ℓ and contains P.

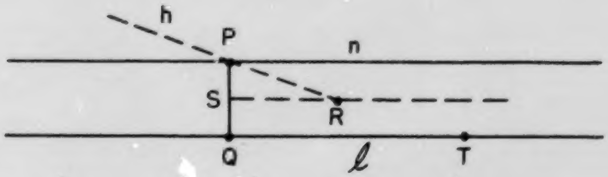
Definition. Two lines are said to be parallel if and only if they are coplanar and do not intersect.

Theorem. If ℓ is a line and P is a point not on ℓ, then there is at least one line parallel to ℓ through P.

Proof: Using the notation of the theorem, there is a line m through P perpendicular to ℓ. There is also a line n perpendicular to m at P in the plane determined by P and ℓ. Line n is thus coplanar with ℓ. Furthermore, n contains P and does not intersect ℓ, for, if it did intersect ℓ, the Angle Sum Theorem would be contradicted. Hence n is parallel to ℓ through P.

Theorem. If ℓ is a line and P is a point not on ℓ, then there is at most one line parallel to ℓ through P .

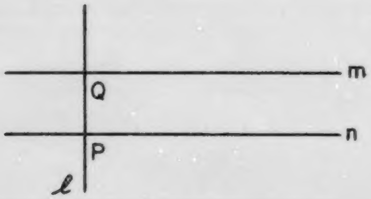
Proof: Using the notation of the theorem, from the proof of the previous theorem we know that there is a line n through P parallel to ℓ, and that the perpendicular to ℓ through P that meets ℓ at Q is such that $\overleftrightarrow{PQ}$ is perpendicular to both ℓ and n .



Now suppose that there is a line h through P that is parallel to ℓ and distinct from n . Since h is distinct from n and parallel to ℓ, there is a point R of h that lies in the intersection of the halfplane determined by ℓ and P and the halfplane determined by n and Q . The perpendicular through R to $\overleftrightarrow{PQ}$ meets $\overleftrightarrow{PQ}$ in a point S between P and Q . Let k be the number such that $PQ = kPS$. We see that $k \neq 0$. In the halfplane determined by PQ and R there is a point T on ℓ such that $QT = kRS$. By the S.A.S. Similarity Postulate, $\Delta PSR \sim \Delta PQT$. Hence $\angle SPR \cong \angle QPT$. Therefore $\overleftrightarrow{PQ}$ coincides with $\overleftrightarrow{PT}$, that is, h intersects ℓ at T . This contradiction to the assumption that h is parallel to ℓ and different from n leads to the conclusion that there is at most one parallel to ℓ through P .

Theorem. If ℓ is coplanar with two parallel lines and perpendicular to one of them, then it is perpendicular to the other.

Proof: Let the two parallel lines be denoted by m and n. Suppose that ℓ is perpendicular to n. Let P denote the intersection of ℓ and n. Since there is one and only one parallel to m through P, ℓ cannot be parallel to m. Since ℓ is coplanar with m it therefore intersects m in a point Q.



As in the proof of existence of line parallel to a given line through a point not on the given line, the line through Q in the plane of n and Q and perpendicular to ℓ is parallel to n. Since there is only one parallel to n through Q, this line must be m. Hence m is perpendicular to ℓ.

Definition. A transversal of two coplanar lines is a line that intersects their union in two points.

Definition. An angle contained in the union of two coplanar lines and a transversal of them is said to be an interior angle formed by the transversal and the pair of lines if and only if a side of the angle is in one of the lines and the other side of the angle is in the transversal and contains a point of the second line.

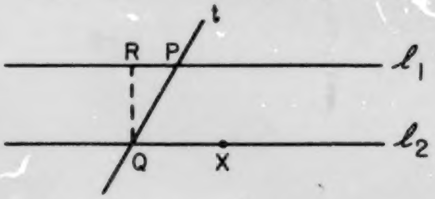
Definition. Two interior angles formed by a transversal to a pair of coplanar lines are said to be alternate interior angles if and only if the intersection of the interiors of the angles is empty, and they do not form a linear pair.

Theorem. If two alternate interior angles formed by a transversal to a pair of coplanar lines are congruent, then the other two alternate interior angles formed are also congruent.

Proof: Based on fact that supplements of angles with equal measure have equal measure.

Theorem. If ℓ_1 and ℓ_2 are parallel lines and t is a transversal to them then alternate interior angles are congruent.

Proof: Let the points of the intersection of t with ℓ_1, ℓ_2 be P and Q, respectively.

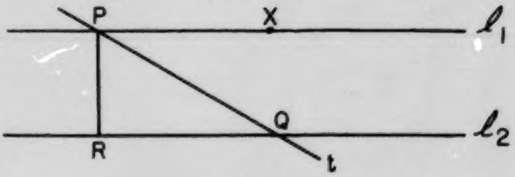


The perpendicular through Q to ℓ_1 meets ℓ_1 in a point R. Since ℓ_1 and ℓ_2 are parallel, $\overleftrightarrow{QR}$ is therefore perpendicular to ℓ_2. If R coincides with P, then alternate interior angles are right angles and are congruent. If R does not coincide with P, then $m\angle QPR + m\angle PQR = 90$, since QRP is a right triangle. If X is a point on ℓ_2 in the halfplane determined by t and not containing R, then $m\angle PQX + m\angle PQR = 90$. Hence $\angle QPR \cong \angle PQX$. Thus alternate interior angles are congruent.

Theorem. If a transversal t to a pair of coplanar lines ℓ_1 and ℓ_2 is such that the alternate interior angles formed are congruent, then ℓ_1 and ℓ_2 are parallel.

Proof: Use the notation of the theorem. Let P and Q denote the points of t on ℓ_1 and ℓ_2, respectively. If t is perpendicular to both ℓ_1 and ℓ_2, then ℓ_1 and ℓ_2 cannot intersect without contradicting the Angle Sum Theorem, and are therefore parallel, since they are coplanar.

If t is not perpendicular to both ℓ_1 and ℓ_2, then it is perpendicular to neither of them. Let R be the point of intersection of the perpendicular through P to ℓ_2.



Let X be a point on ℓ_1 on the opposite side of t from R. By hypothesis $m\angle XPQ = m\angle PQR$. But, since $\angle PRQ$ is a right angle, $m\angle RPQ + m\angle PQR = 90$. Hence $m\angle RPQ + m\angle XPQ = 90$. Therefore $\overline{PR}$ is perpendicular to both lines ℓ_1 and ℓ_2. Therefore ℓ_1 and ℓ_2 cannot meet without contradicting the Angle Sum Theorem. Since ℓ_1 and ℓ_2 are coplanar, they are therefore parallel.

6. Pythagorean Theorem

Definitions. In a right triangle the side that is not contained in the right angle is called the hypotenuse.

The sides that are contained in the right angle are called the legs.

Theorem. (Pythagorean Theorem.)

If a triangle is a right triangle the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs.

Proof: As in the text, or any of a variety of proofs in the literature.

Theorem. (Converse of the Pythagorean Theorem.)

If the sum of the squares of the lengths of two sides of a triangle is equal to the square of the length of the third, then the triangle is a right triangle and the third side is the hypotenuse.

Proof: As in the text, or any of a variety of proofs in the literature.

INTRODUCTION TO NON-EUCLIDEAN GEOMETRY

About one hundred and fifty years ago, a revolution in mathematical thought began with the discovery of a geometrical theory which differed from the classical theory of space formulated by Euclid about 300 B.C. Euclid's Geometry Text, the Elements, was the finest example of deductive thinking the human race had known, and had been so considered for two thousand years. It was believed to be a perfectly accurate description of physical space, and at the same time, the only way in which the human mind could conceive space. It is no small wonder then that the development of theories of non-Euclidean geometry had an impact on mathematical thought comparable to that of Darwin in biology, Copernicus in astronomy or Einstein in physics.

How did this revolutionary change come about? Strangely enough it may be considered to have had its origin in Euclid's text. Although he lists his postulates at the beginning, he refrains from employing one of them until he can go no further without it. This is the famous fifth postulate which we may state in equivalent form as

Euclid's Parallel Postulate. If point P is not on line L, there exists only one line through P which is parallel to L.

It seems probable that Euclid deferred the introduction of the fifth postulate because he considered it more complex and harder to grasp than his other postulates.

The consequences of introducing Euclid's Parallel Postulate are almost phenomenal. Using it we get:

1. The Alternate Interior Angle Theorem for parallel lines,

2. The sum of the measures of the angles of a triangle is 180,

3. Parallel lines are everywhere equidistant,

4. The existence of rectangles of preassigned dimensions.

As remote but recognizable consequences of Euclid's Parallel Postulate, we have:

5. The familiar theory of area in terms of square units which in effect reduces any plane figure to an equivalent rectangle,

6. The familiar theory of similarity,

7. The Pythagorean Theorem.

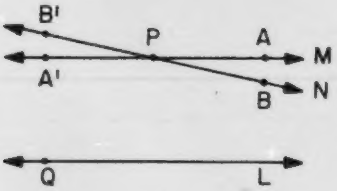
There is no explicit evidence that Euclid considered the fifth postulate an improper assumption in his basis for geometry. But generations of mathematicians for over 2000 years were dissatisfied with it, and worked hard and long in attempts to deduce it as a theorem from the other seemingly simpler postulates. Right up to the beginning of the 19th century able mathematicians convinced themselves that they had settled the problem only to have flaws discovered in their work. Sometimes they employed the principle of the indirect method and developed elaborate and subtle arguments to prove that the denial of Euclid's Parallel Postulate would force one into a contradiction. None of these arguments stood up under analysis. Finally early in the 19th century, J. Bolyai (1802-1860), a Hungarian army officer, and N. I. Lobachevsky (1793-1856), a Russian professor of mathematics at the University of Kazan, independently introduced theories of geometry based on a contradiction of Euclid's Parallel Postulate.

The purpose of this talk is to give an elementary introduction to the non-Euclidean theory of geometry which Bolyai and Lobachevsky created.

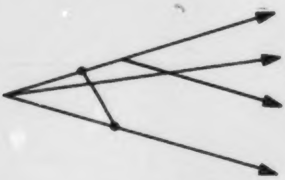
I. Two Non-Euclidean Theorems

In this part we try to give you, without a long preliminary discussion, the flavor of non-Euclidean geometry. Our viewpoint is this: Suppose we consider the hypothesis that there are two lines parallel to a particular line through a particular point. What will follow? As a basis for our deductions we assume the postulates of Euclidean geometry except for the Parallel Postulate, specifically Postulates 1 through 21 of the text.

Theorem 1. Let P be a point and L a line not containing P such that there are two distinct lines through P each of which is parallel to L . Then L is wholly contained in the interior of some angle.



Proof: Let lines M and N contain P and be parallel to L . Then M and N separate the plane into four "parts" each of which is the interior of an angle. Specifically these parts or regions may be labeled as the interiors of the angles $\angle APB$, $\angle A'PB'$, $\angle A'PB$, $\angle APB'$ where P is between A and A' on M and P is between B and B' on N . Let Q be any point of L . Since L does not meet M or N , Q is not on M or N . So Q is in one of the four angle interiors, say the interior of $\angle A'PB$. Now where can L lie? Note that one of its points Q is in the interior of $\angle A'PB$ and that L does not meet the sides of the angle $\angle A'PB$. Clearly L is "trapped" inside $\angle A'PB$ and the theorem is proved.



Observe how strange this is when compared with the Euclidean situation where only a part of a line can be contained in the interior of an angle, as indicated in the figure. But note--as always in mathematics--the inevitability of the result once the hypothesis is granted. You may say the argument is valid abstractly--but it doesn't correspond to physical reality.

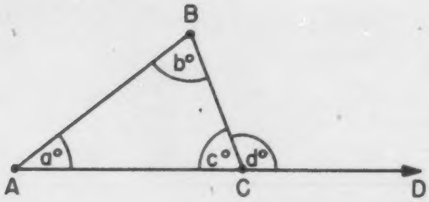
As you make a statement like this you begin to tread the path of the non-Euclidean geometers. All that one needs to think mathematically is a set of precisely stated assumptions (postulates) from which conclusions (theorems) can be derived by logical reasoning. Are these assumptions absolutely true when applied to the physical world? We don't really know. It is not our professional concern as mathematicians to answer the question. It lies in the domain of physicists, astronomers and surveyors. As human beings who work in mathematics we may like to feel that our theories are applicable to physical reality. But this doesn't require the absolute truth of our postulates or our theorems. When Euclidean geometry is applied by an architect or engineer or surveyor he doesn't require results which are absolutely correct--he might consider this a mirage. Rather he demands results correct to the degree of precision required by his problem--accuracy of one part in a hundred might be excellent in a pocket magnifying glass but one part in a million might be too rough for a far-ranging astronomical telescope.

Our first theorem indicated how positional or non-metrical properties in a non-Euclidean geometry might differ from our Euclidean expectations. Now we show how metrical properties --specifically the angle sum of a triangle--are altered when we change the Parallel Postulate.

Theorem 2. Let P be a point and L a line such that there are two lines through P each of which is parallel to L . Then there exists at least one triangle the sum of whose angle measures is less than 180 .

We first prove a lemma.

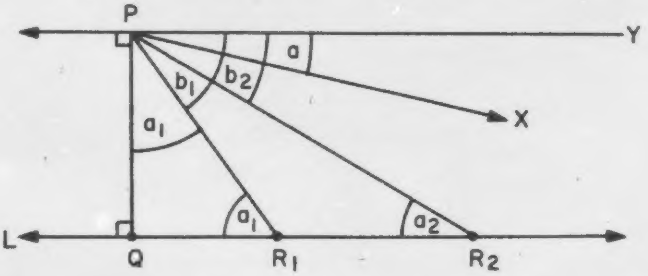
Lemma. If the sum of the angle measures of a triangle is greater than or equal to 180 then the measure of an exterior angle is less than or equal to the sum of the measures of the two remote interior angles.



Proof: We have $a + b + c \geq 180$.

Hence $a + b \geq 180 - c = d$.

Proof of Theorem 2: Suppose the theorem false. Then the sum of the angle measures of every triangle is greater than or equal to 180 .



Let L be a line and P a point not on L such that there are two lines through P parallel to L . Let line $\overleftrightarrow{PQ}$ be perpendicular to L at Q . Since there are two lines through P parallel to L one of these must make an acute angle with line $\overleftrightarrow{PQ}$. Suppose then line $\overleftrightarrow{PX}$ is parallel to L and makes an acute angle, $\angle QPX$, with line $\overleftrightarrow{PQ}$. Let line $\overleftrightarrow{PY}$ be perpendicular to line $\overleftrightarrow{PQ}$ with Y on the same side of line $\overleftrightarrow{PQ}$ as X . Let $m \angle YPX = a$; then $a < 90$. (Think of a as a small positive number, say .1 .) Now there is a point R_1 on L such that $QR_1 = PQ$ and R_1 is on the same side of $\overleftrightarrow{PQ}$ as X and Y . Then ΔPQR_1 is isosceles so that $m \angle QPR_1 = m \angle QR_1P = a_1$. Since the exterior angle of ΔPQR_1 at Q is a right angle, the Lemma implies

$$a_1 + a_1 = 2a_1 \geq 90$$

and

$$a_1 \geq 45 .$$

Let $m\angle YPR_1 = b_1$. Then

$$b_1 + a_1 = 90 ,$$

so that

$$b_1 = 90 - a_1$$

and

$$b_1 \leq 45 .$$

Moreover

$$b_1 > a .$$

Now we repeat the argument with a new triangle. On ray $\overrightarrow{QR_1}$, there is a point R_2 such that $R_1R_2 = PR_1$ and R_1 is between Q and R_2 . Then ΔPR_1R_2 is isosceles, so that $m\angle R_1PR_2 = m\angle R_1R_2P = a_2$. By the Lemma

$$a_2 + a_2 = 2a_2 \geq a_1 .$$

So that

$$2a_2 \geq a_1 \geq 45$$

and

$$a_2 \geq \frac{45}{2} .$$

Let $m\angle YPR_2 = b_2$. Then

$$b_2 + a_2 = b_1 ,$$

$$b_2 = b_1 - a_2 .$$

Since $b_1 \leq 45$ and $a_2 \geq \frac{45}{2}$ we have

$$b_2 \leq \frac{45}{2} .$$

Moreover

$$b_2 > a .$$

Continuing in this way we obtain a sequence of real numbers

$$b_1 , b_2 , b_3 , \ldots ,$$

which are less than or equal to, respectively,

$$45 , \frac{45}{2} , \frac{45}{4} , \ldots ,$$

all of which are greater than the fixed positive number a . This is impossible since repeated halving of 45 must eventually produce a number less than a . So our supposition is false and the theorem holds.

A proof of this type, though not difficult, may be unfamiliar and you may have to mull it over a bit to appreciate it better. In intuitive terms it is not very hard. There are two main points. First, the ray $\overrightarrow{PX}$ which doesn't meet L acts as a sort of boundary for the rays $\overrightarrow{PR_1}$, $\overrightarrow{PR_2}$, ... which do meet L . Thus the angles $\angle YPR_1$, $\angle YPR_2$, ... have measures b_1 , b_2 , ... which are greater than a . On the other hand (if the sum of the angle measures of every triangle is at least 180) we can "pile up" successive angles $\angle QPR_1$, $\angle R_1PR_2$, ..., starting [illegible] ray $\overrightarrow{PQ}$, of measures *at least* 45 , $\frac{45}{2}$, $\frac{45}{4}$, ..., so that the angles $\angle YPR_1$, $\angle YPR_2$, ..., have measures *at most* 45 , $\frac{45}{2}$, $\frac{45}{4}$, ... So we have a contradiction in that the angles $\angle YPR_1$, $\angle YPR_2$, ..., have measures which approach zero but are all greater than a fixed positive number a .

A final remark. You may object that we have not really justified that $\overrightarrow{PX}$ is a "boundary" for $\overrightarrow{PR_1}$, $\overrightarrow{PR_2}$, ... To take care of this observe that the interiors of $\overrightarrow{PR_1}$ and $\overrightarrow{PX}$ are on the same side of line $\overleftrightarrow{PQ}$. Consequently one of them must fall inside the angle formed by $\overrightarrow{PQ}$ and the other. Suppose the interior of $\overrightarrow{PX}$ fell inside $\angle QPR_1$. Then $\overrightarrow{PX}$ would meet line $\overleftrightarrow{QR_1}$. Since this is impossible, the interior of $\overrightarrow{PR_1}$ must lie inside $\angle QPX$. Similarly for $\overrightarrow{PR_2}$, ...

II. Neutral Geometry

We are using the term "neutral geometry" in this part to indicate that we are assuming neither Euclid's Parallel Postulate nor its contradiction. We shall merely deduce consequences of Euclid's Postulates other than the Parallel Postulate, (specifically our discussions are based on Postulates 1 through 21 of the text). Our results then will hold in Euclidean Geometry and in the non-Euclidean geometry of Bolyai and Lobachevsky since they are deducible from postulates which are common to both theories. Our study is neutral also in the sense of avoiding controversy over the Parallel Postulate. Actually its study helps us to accept the idea of non-Euclidean geometry since it points up the fact that mathematically we have a more basic geometrical theory which can be completed in either of two ways.

We proceed to derive some results in neutral geometry. Since you are familiar with so many striking and important theorems which do depend on Euclid's Parallel Postulate you might think that there are no interesting theorems in neutral geometry. However, this is not so. First we restate as Theorem 3 of this Talk, Theorem 5-10 of the Text.

Theorem 3. The measure of an exterior angle of a triangle is greater than the measure of either of its non-adjacent interior angles.

This is a theorem in "neutral geometry" since it does not depend on the Parallel Postulate. It has some interesting corollaries.

Corollary 1. The sum of the measures of two angles of a triangle is less than 180.

Proof: Given $\triangle ABC$ we show $m\angle A + m\angle B < 180$. By the theorem $m\angle A$ is less than the measure of an exterior angle at B which is $180 - m\angle B$. Thus

$$m\angle A < 180 - m\angle B$$

so that

$$m\angle A + m\angle B < 180 .$$

This corollary is important since, without assuming a parallel postulate, it gives us information about the angles of a triangle. It tells us for example, that a triangle can have at most one obtuse angle or at most one right angle.

Corollary 2. In a plane two lines are parallel if they are both perpendicular to the same line (compare Text, Theorem 6-1).

Proof: The basic properties of perpendicular lines in Euclidean geometry are studied prior to the introduction of the Parallel Postulate, and so are part of (or are valid in) neutral geometry. Thus the familiar proof of the corollary is applicable: If the two lines met we would have, in a plane, two lines perpendicular to the same line through the same point. This is impossible since we would then have a triangle with the measure of an exterior angle equal to that of a non-adjacent interior angle. Hence the lines can't meet.

Corollary 3. Let L be a line, and let P be a point not on L. Then there is at least one line through P, parallel to L.

Proof: This follows from Corollary 2 by the familiar theorem on the existence of perpendiculars. (See Text, Theorem 4-21 and 5-11) Let L_1 be the unique line which contains P and is perpendicular to L. Let L_2 be the unique line which contains P, is perpendicular to L_1, and is coplanar with L.

Observe that this familiar--almost hackneyed--discussion has yielded a very important principle: That parallel lines exist. More precisely, there exists at least one line parallel without assuming any parallel postulate! So the crucial point in our study of the theory of parallelism is whether there is one, or more than one, line parallel to a given line through an external point.

To prove an important, and not sufficiently well known, theorem of Legendre (1752-1833) we introduce the following:

Lemma. Given ΔABC and $\angle A$. Then there exists a triangle $\Delta A_1B_1C_1$ such that: (a) it has the same angle measure sum as ΔABC; (b) $m\angle A_1 \leq \frac{1}{2} m\angle A$.

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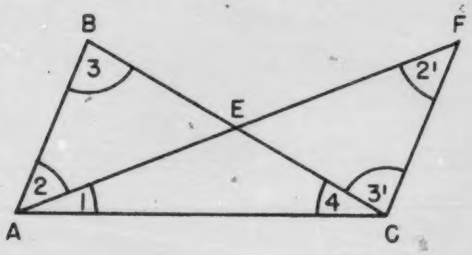

Proof: We use the same configuration as in Theorem 3 (see the proof of Theorem 5-10 in Text). Let E be the midpoint of $\overline{BC}$ and let F satisfy AE = EF and E is between A and F . Then $\Delta BEA \cong \Delta CEF$ and corresponding angles have equal measures. ΔAFC is the $\Delta A_1B_1C_1$ we are seeking. We have

$$\begin{aligned} m\angle A + m\angle B + m\angle C &= m\angle 1 + m\angle 2 + m\angle 3 + m\angle 4 \\ &= m\angle 1 + m\angle 2' + m\angle 3' + m\angle 4 \\ &= m\angle CAF + m\angle AFC + m\angle FCA\ . \end{aligned}$$

To complete the proof note that

$$m\angle A = m\angle 1 + m\angle 2 = m\angle 1 + m\angle 2'$$

so that

$$m\angle A = m\angle CAF + m\angle AFC\ .$$

Hence one of the terms on the right is less than or equal to $\frac{1}{2}$ the term on the left, that is $\frac{1}{2}m\angle A$. Consequently ΔAFC can be relabeled $\Delta A_1B_1C_1$ so as to make the theorem valid.

Note that since we have not assumed Euclid's Parallel Postulate we don't know that the angle measure sum is constant for all triangles. So the lemma is a significant result in that we can construct from a given triangle a new one with the same angle measure sum. In intuitive terms we can replace a triangle by a "slenderer" one without altering its angle measure sum. In effect the proof shows this by cutting off ΔABE from ΔABC and pasting it back on as ΔFCE .

Now we can prove the following remarkable theorem.

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Theorem 4. (Legendre.) The angle measure sum of any triangle is less than or equal to 180 .

Proof: Suppose the contrary. Then there must exist a triangle ΔABC , whose angle measure sum is $180 + p$, where p is a positive number. Now we apply the Lemma. It tells us that there exists a slenderer triangle, $\Delta A_1B_1C_1$, whose angle measure sum also is $180 + p$ such that $m \angle A_1 \leq \frac{1}{2} m \angle A$. To fix our ideas let us say $p = 1$ and $m \angle A = 25$. Then

$$m \angle A_1 + m \angle B_1 + m \angle C_1 = 181 \quad \text{and} \quad m \angle A_1 \leq \frac{25}{2} .$$

Pressing our advantage we reapply the lemma. So there is a still slenderer triangle, let us call it $\Delta A_2B_2C_2$, whose angle measure sum is $180 + p$ and $m \angle A_2 \leq \frac{1}{2} m \angle A_1$. That is,

$$m \angle A_2 + m \angle B_2 + m \angle C_2 = 181 \quad \text{and} \quad m \angle A_2 \leq \frac{25}{4} .$$

Continuing in this way, we get a sequence of triangles each with angle measure sum 181 and with successive angles of measures no greater than

$$25 , \frac{25}{2} , \frac{25}{4} , \frac{25}{8} , \ldots$$

To see this is impossible, consider $\Delta A_5B_5C_5$ for which $m \angle A_5 < p$.

We have

$$m \angle A_5 + m \angle B_5 + m \angle C_5 = 181 \quad \text{and} \quad m \angle A_5 \leq \frac{25}{32} .$$

Certainly

$$m \angle A_5 < 1 ,$$

but

$$m \angle B_5 + m \angle C_5 < 180$$

by Corollary 1 to Theorem 3. Adding the inequalities,

$$m \angle A_5 + m \angle B_5 + m \angle C_5 < 181 .$$

This contradiction implies our supposition false, and the theorem is established.

Note the point of the proof is to get a triangle so "slender," that is with one angle so small, that the triangle can't exist by Corollary 1 above. It may now be instructive to write out the proof in general terms without assigning specific values to p and $m \angle A$.

Corollary 4. The angle measure sum of any quadrilateral is less than or equal to 360.

III. Do Rectangles Exist?

We continue to study neutral geometry. We are interested in whether a rectangle can exist in such a geometry, and what happens if it does. So most of our theorems will have the hypothesis that a rectangle exists. We use freely the results of Part II on neutral geometry.

The existence of a rectangle in a geometry is not a trivial thing--imagine what Euclidean geometry would be like if you didn't have or couldn't use rectangles. If you try to "construct" a rectangle you will find you are assuming Euclid's Parallel Postulate or one of its consequences, such as, the angle measure sum of a triangle is 180.

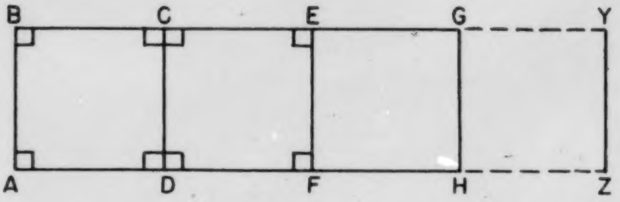
First, we formally define rectangle as we shall use the term. Our definition differs from that of the text (a rectangle is a parallelogram with one right angle); the text definition is unsuitable for our purposes here.

Definition. A (plane) quadrilateral is called a rectangle if each of its angles is a right angle.

Notice that since we are operating in neutral geometry and have not assumed Euclid's Parallel Postulate, we can't automatically apply familiar Euclidean propositions, such as (1) the opposite sides of a rectangle are parallel, or (2) that they are equal in length, or (3) that a diagonal divides a rectangle into two congruent triangles. If we want to assert any of these results we will have to prove them from our definition without assuming a parallel postulate. For example, (1) is immediate by Corollary 2.

Theorem 5. If one particular rectangle exists then a rectangle exists with an arbitrarily large side.

Restatement: Suppose a rectangle $ABCD$ exists and x is a given positive real number. Then there exists a rectangle with one side of length greater than x.



Proof: On ray $\overrightarrow{BC}$ there is a point E such that $BC = CE$ and C is between B and E. Similarly, on $\overrightarrow{AD}$ there is a point F such that $AD = DF$ and D is between A and F. In order to show $ABEF$ is a rectangle, we need to show that $\angle BEF$ and $\angle AFE$ are right angles. Now $\Delta DCB \cong \Delta DCE$ by S.A.S., since $DC = DC$, $CB = CE$, and the two angles at C are right angles. Thus $BD = DE$ and $\angle CDB \cong \angle CDE$. Since $\angle CDB \cong \angle CDE$, we have $\angle BDA \cong \angle EDF$. Therefore, since in addition $DA = DF$, we have by S.A.S. that $\Delta DAB \cong \Delta DFE$. Since $\angle DAB \cong \angle DFE$ and $m \angle BAD = 90$, we conclude that $\angle AFE$ is a right angle. Similarly, we can show that $\angle BEF$ is a right angle. Hence $ABEF$ is a rectangle such that

$$AF = 2AD .$$

Continuing in the same manner, we show that $ABGH$ is a rectangle and that

$$AH = 3AD .$$

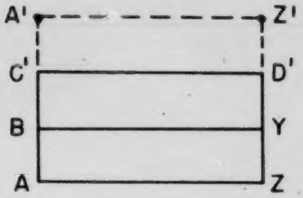
Indeed, it is clear that by application of this idea we can prove the existence of a rectangle $ABYZ$ such that

$$AZ = n \cdot AD$$

for each positive integer n. Now choose n so big that $n \cdot AD > x$. Then $ABYZ$ satisfies the conditions of our theorem.

Corollary 5. If one particular rectangle exists, then a rectangle exists with two arbitrarily large adjacent sides.

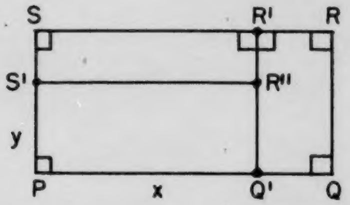
Restatement: Suppose a rectangle $ABCD$ exists and x, y are given positive real numbers. Then there exists a rectangle $PQRS$ such that $PQ > y$ and $PS > x$.



Proof: By Theorem 5, there is a rectangle $ABYZ$ with $AZ > x$. By the method used in the proof of Theorem 5, there is a rectangle $AC'D'Z$ such that $AC' = 2AB$. Continuing as in the proof of Theorem 5, we prove that there exists a rectangle $AA'Z'Z$ with $AA' > y$. Since $AZ > x$, we have shown that a rectangle with two arbitrarily large adjacent sides exists.

Theorem 6. If one particular rectangle exists then a rectangle exists with two adjacent sides of preassigned lengths x, y .

Proof: By the last corollary there is a rectangle $PQRS$ such that $PQ > x$ and $PS > y$.

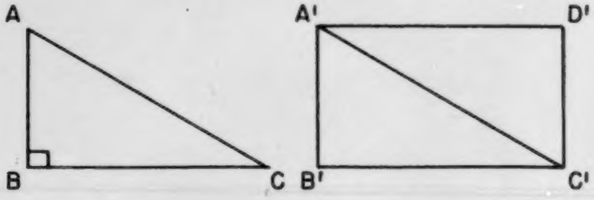


There is a point Q' on $\overline{PQ}$ such that $PQ' = x$. There is a perpendicular from Q' to line $\overleftrightarrow{RS}$ with foot R' . We show $PQ'R'S$ is a rectangle. It certainly has right angles at P, S, R' . We show $\angle PQ'R'$ also is a right angle. Suppose $m \angle PQ'R' > 90$. Then the sum of the angle measures of quadrilateral $PQ'R'S$ is greater than 360 contrary to the corollary of Legendre's Theorem (Part II). Suppose

$m\angle PQ'R' < 90$. Then $m\angle QQ'R' > 90$ and quadrilateral $QQ'R'R$ has an angle measure sum greater than 360 . Thus the only possibility is $m\angle PQ'R' = 90$, and $PQ'R'S$ is a rectangle.

In the same way there is a point S' in $\overline{PS}$ such that $PS' = y$. There is a perpendicular from S' to line $\overleftrightarrow{Q'R'}$ with foot R'' . Then as above $PQR''S'$ is a rectangle, and it has sides $\overline{PQ'}$ and $\overline{PS'}$ of lengths x and y .

Theorem 7. If one particular rectangle exists then every right triangle has an angle measure sum of 180 .

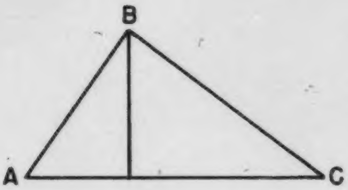


Proof: Our procedure is to show: (1) any right triangle is congruent to a triangle formed by the splitting of a rectangle by a diagonal, and (2) the latter type of triangle must have an angle measure of 180 . Let ΔABC be a right triangle with right angle at B . By Theorem 6 there exists a rectangle $A'B'C'D'$ with $A'B' = AB$ and $B'C' = BC$. Then $\Delta ABC \cong \Delta A'B'C'$ and they have the same angle measure sum. Let p be the angle measure sum of $\Delta A'B'C'$ and q be that of $\Delta A'C'D'$. We have

(1) $$p + q = 4 \cdot 90 = 360 .$$

We want to show $p = 180$. By Legendre's Theorem $p < 180$ or $p = 180$. Suppose $p < 180$. Then by (1) $q > 180$, contrary to Legendre's Theorem. So $p = 180$ must hold and the proof is complete.

Theorem 8. If one particular rectangle exists then every triangle has an angle measure sum of 180 .



Proof: Any triangle ΔABC can be "split" into two right triangles. Each of these has angle measure sum 180 by Theorem 7. It easily follows that the same holds for ΔABC .

This is a rather striking result: The existence of one puny rectangle with microscopic sides inhabiting a remote portion of space guarantees that every conceivable triangle has an angle measure sum of 180 . Since this is a typically Euclidean Property we are tempted to say that if in a neutral geometry a rectangle exists, the geometry must be Euclidean. The statement is correct but not fully justified, since to characterize a neutral geometry as Euclidean we must know that it satisfies Euclid's Parallel Postulate. This can now be proved without trouble.

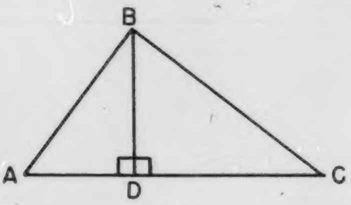
Theorem 9. If one particular rectangle exists then Euclid's Parallel Postulate holds.

Proof: Suppose a rectangle exists but Euclid's Parallel Postulate fails. Then there must exist a line L and a point P such that there are two lines through P parallel to L , since by Corollary 3 there is at least one line parallel to a given line through an external point. Then by Theorem 2 there exists one triangle, at least, whose angle measure sum is less than 180 . This contradicts Theorem 8. Consequently Euclid's Parallel Postulate must hold.

What we have justified is a remarkable equivalence theorem, namely: Euclid's Parallel Postulate is logically equivalent to the existence of a rectangle. That is, taking either of these statements as a postulate we can deduce the other as a theorem, provided of course we assume the postulates for a neutral geometry.

An interesting condition equivalent to the existence of a rectangle is the existence of a triangle whose angle measure is 180 :

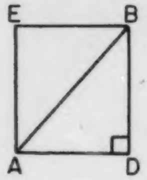
Theorem 10. If there exists one particular triangle with angle measure sum of 180 , then there exists a rectangle.



Proof: Suppose ΔABC has angle measure sum 180 . First we show there is a right triangle with angle measure sum 180 . Split ΔABC into two right triangles, whose angle measure sums are say p and q . Then

$$p + q = 180 + 2 \cdot 90 = 360 .$$

We show $p = 180$. By Legendre's Theorem, $p \leq 180$. If $p < 180$ then $q > 180$ contrary to Legendre's Theorem. Thus $p = q$ and there is a right triangle, say ΔABD , which has angle measure sum 180 .



Now in the plane of ΔABD on the line perpendicular to $\overleftrightarrow{AD}$ at A there is a point E on the same side of $\overleftrightarrow{AD}$ as B and such that $AE = DB$. Since $m \angle EAB = 90 - m \angle BAD = m \angle ABD$, it follows by S.A.S. that $\Delta AEB \cong \Delta BDA$. Hence $\angle AEB$ is a right angle, as is also $\angle DBE$. Therefore $AEBD$ is a rectangle.

Corollary 6. If one particular triangle has angle measure sum 180 then every triangle has angle measure sum 180 .

Proof: By Theorems 10 and 8 .

Corollary 7. If one particular triangle has angle measure sum 180 then Euclid's Parallel Postulate holds.

Proof: By Theorems 10 and 9 .

Corollary 8. If one particular triangle has an angle measure sum which is less than 180 then *every* triangle has an angle measure sum less than 180 .

Proof: Suppose ΔABC has angle measure sum less than 180 . Consider any triangle ΔPQR . By Legendre's Theorem its angle measure sum p must satisfy $p = 180$ or $p < 180$. Suppose $p = 180$. Then by Corollary 6, ΔABC has angle measure sum 180 , contrary to hypothesis. Thus $p < 180$.

Comparing Corollaries 6 and 8 we observe an important fact. A neutral geometry is "homogeneous" in the sense that all of its triangles have an angle measure sum of 180 or they all have angle measure sums less than 180 . The first type of neutral geometry is Euclidean geometry--the second type corresponds to the non-Euclidean geometry developed by Bolyai and Lobachevsky. This will be discussed in the next part.

Exercise 1. Suppose there is only one line parallel to a particular line L through a particular point P . Prove that Euclid's Parallel Postulate holds.

Exercise 2. Suppose there are two lines parallel to a particular line L through a particular point P . Prove there are two lines parallel to each line through each external point.

IV. Lobachevskian Geometry

Now we introduce the non-Euclidean geometry of Bolyai and Lobachevsky as a formal theory based on its own postulates. We call the theory Lobachevskian geometry to signalize the lifetime of work which Lobachevsky devoted to the theory. To study Lobachevskian geometry we merely assume the postulates of Euclidean geometry but replace Euclid's Parallel Postulate by Lobachevsky's Parallel Postulate: If point P is not on line L there are at least two lines through P which are parallel to L . In other words we assume the postulates of neutral geometry (Postulates 1 through 21 of the text) and adjoin Lobachevsky's Parallel Postulate. Consequently the theorems which we have already derived are valid in Lobachevskiar geometry. In fact, by putting together two earlier results we get the following important theorem.

Theorem 11. The angle measure sum of any triangle is less than 180 .

Proof: By Theorem 2 there exists a triangle whose angle measure sum is less than 180 . Hence the same is true of every triangle by Corollary 8.

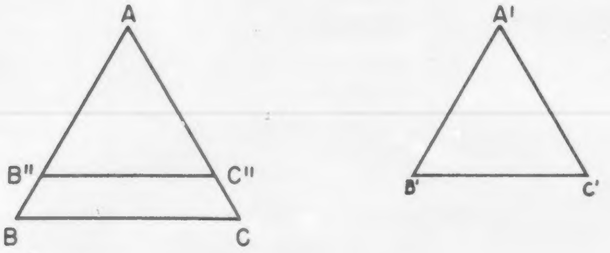
Corollary 9. The angle measure sum of any quadrilateral is less than 360 .

Proof: By the corollary to Legendre's Theorem (Part II, Theorem 2) the only other possibility for the value is 360 , and this is ruled out by Theorem 11.

Corollary 10. There exist no rectangles.

Now we show that similar triangles can't exist in Lobachevskian geometry, except of course for the trivial case of congruent triangles.

Theorem 12. Two triangles are congruent if their corresponding angles have equal measures.



Proof: Suppose the theorem false. Then there exist ΔABC and $\Delta A'B'C'$ which are not congruent such that $m\angle A = m\angle A'$, $m\angle B = m\angle B'$, $m\angle C = m\angle C'$. Since the triangles are not congruent $AB \neq A'B'$ (otherwise they would be congruent by A.S.A.). Similarly $AC \neq A'C'$ and $BC \neq B'C'$. Consider the triples AB, AC, BC and $A'B'$, $A'C'$, $B'C'$. One of these triples must contain two numbers which are greater than the corresponding numbers of the other triple. Consequently it is not restrictive to suppose $AB > A'B'$ and $AC > A'C'$.

Then we can find B'' on $\overline{AB}$ such that $A'B' = AB''$ and C'' on $\overline{AC}$ such that $A'C' = AC''$. It follows that $\Delta AB''C'' \cong \Delta A'B'C'$ so that

$$m\angle AB''C'' = m\angle B' = m\angle B .$$

Hence $\angle BB''C''$ is supplementary to $\angle B$. Similarly $\angle CC''B''$ is supplementary to $\angle C$. Therefore quadrilateral $BB''C''C$ has an angle measure sum of 360. This contradicts Corollary 9 and our proof is complete.

We have here a striking contrast with Euclidean geometry. In view of Theorem 12, in Lobachevskian geometry there cannot be a theory of similar figures based on the usual definition. For if two triangles were similar, the measures of their corresponding angles would be equal and they would have to be congruent. In general two similar figures would be congruent and so have the same size. In a Lobachevskian world, pictures and statues would have to be life-size to avoid distortion.

Now let us consider the question of measurement of area. For the sake of simplicity we restrict ourselves to triangles. Clearly the Euclidean procedure of measuring area in terms of square units will not apply since squares don't exist in Lobachevskian geometry. To clarify the problem we ask what are the essential characteristics of area. As a minimum we require:

(1) The area of a triangle shall be a uniquely determined positive real number;

(2) Congruent triangles shall have equal areas;

(3) If a triangle T is split into two triangles T_1 and T_2 then the area of T shall be the sum of the areas of T_1 and T_2 .

It is easy to verify that the familiar formula for the area of a triangle in Euclidean geometry satisfies these conditions.

There is a similar area formula (or area "function") in Lobachevskian geometry but it is most naturally expressed in terms of the angles of a triangle. To state it formally we introduce the

Definition. The *defect* (or *deficiency*) of ΔABC is $180 - (m\angle A + m\angle B + m\angle C)$.

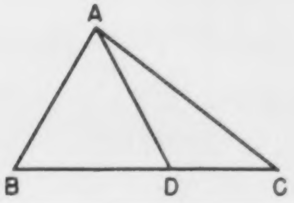
Note that the defect of a triangle literally is the amount by which its angle measure sum falls short of 180 .

The defect of a triangle has the essential properties of area:

Theorem 13. The defect of a triangle satisfies Properties (1), (2), (3), above.

Proof: Clearly (1) is satisfied since the defect of a triangle is a definite positive number. Property (2) holds since congruent triangles have equal angle sums and so equal defects.

To establish (3) let ΔABC be given and let D be a point of $\overline{BC}$, so that ΔABC is split into ΔABD and ΔADC . The sum of the defects of the latter two triangles is



$$\begin{aligned}
&180 - (m\angle BAD + m\angle B + m\angle BDA) + 180 \\
&\quad - (m\angle CAD + m\angle C + m\angle CDA) \\
&= 180 - (m\angle BAD + m\angle CAD + m\angle B + m\angle C) \\
&= 180 - (m\angle BAC + m\angle B + m\angle C)
\end{aligned}$$

which is the defect of ΔABC .

Are there other area functions besides the defect? It is easy to verify that if we multiply the defect by any positive constant k, we obtain an area function which satisfies Properties (1), (2), (3). This is not as remarkable as it might seem, since the specific form of our definition of defect depends on our basic agreement to measure angles in terms of degrees. If we adopt a different unit for the measure of angles and define "defect" in the natural manner, we obtain a constant multiple of the defect as we defined it. To be specific, suppose we change the unit of angle measurement from degrees to minutes. This would entail two simple changes in the above theory: (a) each angle measure would have to be multiplied by 60; (b) the key number 180 would have to be replaced by 60 times 180. Thus the appropriate definition of "defect" would be 60 times the defect as we defined it.

Finally we note that it can be proved that any area function satisfying (1), (2), (3) must be k times the defect (our definition) for some positive constant k. In view of this it is natural to define the area of a triangle to be its defect.

Query. Which of the Properties (1), (2), (3) holds for the defect of a triangle in Euclidean geometry?

It is interesting to note that in Euclidean spherical geometry the sum of the angle measures of a triangle is greater than 180 and the area of a triangle is given by its "excess," that is its angle measure sum minus 180.

Exercise 1. Given ΔABC with points, D, E, F in $\overline{AB}$, $\overline{BC}$, $\overline{AC}$ respectively. Prove that the defect of ΔABC is the sum of the defects of the triangles ADF, BED, CFE, and DEF.

Exercise 2. If points P, Q, R are inside ΔABC prove that ΔABC has a larger defect than ΔPQR.

We conclude this part by observing that the familiar Euclidean property--parallel lines are everywhere equidistant --fails in Lobachevskian geometry. In fact there are parallel lines of two types. If two parallel lines have a common perpendicular they diverge continuously on both sides of this perpendicular. If two parallel lines don't have a common perpendicular they are asymptotic--that is, if a point on one recedes endlessly in the proper direction, its distance to the other will approach zero.

Conclusion

In its further development Lobachevskian geometry is at least as complex as Euclidean geometry. There is a Lobachevskian solid geometry, a trigonometry and an analytic geometry--problems in mensuration of curves, surfaces and solids require the use of the calculus.

You may object that the structure is grounded on sand --that Lobachevskian geometry is inconsistent and eventually will yield contradictory theorems. This of course was the implicit belief that led mathematicians for 2,000 years to try to prove Euclid's Parallel Postulate. Actually we have no absolute test for the consistency of any of the familiar branches of mathematics. But it can be proved that the Euclidean and Lobachevskian geometries stand or fall together on the question of consistency. That is, if either is inconsistent, so is the other.

Once the ice had been broken by Bolyai's and Lobachevsky's successful challenges to Euclid's Parallel Postulate, mathematicians were stimulated to set up other non-Euclidean geometries--that is, geometric theories which contradict one or more of Euclid's Postulates, or approach geometry in an essentially different way. The best known of these was proposed in 1854 by the German mathematician Riemann (1826-1866). Riemann's theory contradicts Euclid's Parallel Postulate by assuming there are no parallel lines. This required the abandonment of other postulates of Euclid since we have proved the existence of parallel lines without assuming any parallel postulate (Corollary 3). In Riemann's theory, in contrast to those of Euclid and Lobachevsky, a line has finite length. Actually there are two types of non-Euclidean geometry associated with Riemann's name, one called single elliptic geometry in which any two lines meet in just one point, and a second, double elliptic geometry, in which any two lines meet in two points. The second type of geometry can be pictured in Euclidean space as the geometry of points and great circles on a sphere.

Riemann also introduced a radically different kind of geometric theory which builds up the properties of space in the large by studying the behavior of distance between points which are close together. This theory, called Riemannian Geometry, is useful in applied mathematics and physics and is the mathematical basis of Einstein's General Theory of Relativity.

Bolyai and Lobachevsky have opened for us a door on a new and apparently limitless domain.

Answers To Illustrative Test Items

Chapter 2

1. (a) a subset of, or not identical with.
   (b) (all) natural numbers
   (c) true
   (d) may be, can be
   (e) true
   (f) at least four
   (g) coplanar
   (h) which does not contain it
   (i) inductively

2. (a) A_1
   (b) A_4
   (c) A_3
   (d) A_5
   (e) A_6

3. (a) 2
   (b) 4
   (c) 3
   (d) 3

4. (a) False
   (b) True
   (c) False
   (d) True
   (e) True
   (f) True
   (g) False
   (h) False
   (i) False
   (j) True

5. B, none, A, P.

6. (a) (If) three points lie in one plane
   (b) (If) a set is empty
   (c) (When) $x = 5$
   (d) The product of two integers, or integers are being multiplied.

7. (a) If a number is greater than zero, then it is a positive number.
   (b) If two lines intersect in a single point, then they are not parallel.

(c) *If the light is red*, then we stop.

(d) *If a line does not lie in a given plane and if that line and plane intersect*, they intersect in only one point.

(e) *If a and b are real numbers and such that $a \neq b$*, then $(a - b)^2$ is positive.

(f) *If a polygon has four sides*, it is called a quadrilateral.

(g) *If a figure is a square*, it is a rectangle.

8. See problem 7.

9. (b) one (c) point, plane (d) point, plane

(e) planes, point (f) two

10. (a) one point (b) point (c) point

(d) line (e) the same line

(f) distinct (g) contradicts, two (more than one)

(h) exactly one

11. Suppose there is one person who is completely bald, and there is just one person with 1 hair, and just one person with 2 hairs, etc. Then there can be only 300,000 people. But there are 500,000 people. Therefore, our supposition is wrong, and there must be two people with the same number of hairs on their heads.

12. Several wordings are acceptable for the theorems.

1. (a) BIMTON contains at least one HUT.

   (b) BIMTON contains at least two HUTS.

   (c) BIMTON contains at least three HUTS.

2. (a) BIMTON contains at least three BIMS.

   (b) BIMTON contains at least three BIMS that are not all in the same HUT.

3. If two distinct HUTS intersect, their intersection contains exactly one BIM.

[Note: This problem can be expanded by adding the line-plane postulates. A substitute for "plane" could be "HILL." In this case, more theorems could be asked for.]

Answers to Illustrative Test Items
Chapter 3

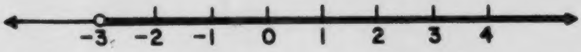
Part I

1. (a) $p > 0$
   (b) $-10 < k < 0$
   (c) $10 < x < 20$
   (d) $r \leq 0$

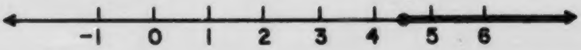
2. (a) $b > c$ Transitive
   (b) $<$ Multiplication by a negative number
   (c) $>$ Addition

3. (a) All numbers x such that $x > -3$.
   (b) All numbers x such that $x \geq \frac{9}{2}$.

4. (a)



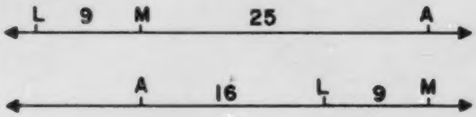
(b)



5. The set of real numbers x such that:
   (a) All x such that $x \geq 0$.
   (b) All x such that $x \leq 1$.
   (c) All x such that $0 \leq x \leq 1$.
   (d) All x such that $x \leq 1$.
   (e) All x such that $x \geq 1$.

6. $|\;|$

7. (a) No.
   (b) Amity.
   (c) 34 miles or 16 miles
   (d)



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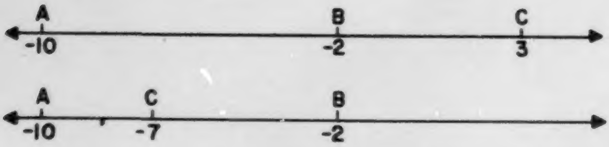

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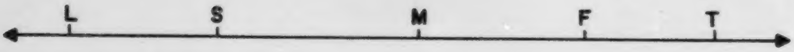

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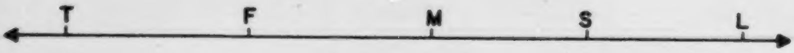


10. (a) $\overleftrightarrow{AB}$ (b) $\overline{MR}$ (c) $\overrightarrow{RQ}$

11. (a) 7
(b) $x + 7$
(c) y
(d) $y - 3$
(e) $y - x$

12. (a) $\frac{1}{3}$ (b) 36

13. (a) $\overline{GK}$
(b) G
(c) $\overline{GH}$
(d) empty set
(e) $\overline{FH}$
(f) $\overline{GK}$
(g) $\overline{GH}$
(h) $\overrightarrow{GK}$ (or $\overrightarrow{GJ}$, or $\overrightarrow{GI}$, or $\overrightarrow{GH}$).

14. (a) ℓ or $\overleftrightarrow{GK}$
(e) $\overrightarrow{FH}$
(g) $\overline{EK}$
(h) $\overrightarrow{GK}$ or $\overrightarrow{GI}$

15. (a) True
(b) Not meaningful
(c) False
(d) True
(e) Not meaningful
(f) Not meaningful
(g) Not meaningful

(h) True
(i) True
(j) True
(k) Not meaningful
(l) False

16. (a) 1 (b) 1 (c) 2 (d) $\frac{1}{2}$ (e) $\frac{2}{9}$

17. (a) 1 (b) $2\frac{1}{2}$ (c) 2 (d) -1 (e) $-2\frac{1}{3}$

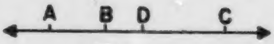
18. (a) 2 (b) 3 (c) 1 (d) 5

19. -2

20. -5

21. (a) 7, -9 (b) 23, -25 (c) -1 (d) 1, -3

22. $AB + BC = AC$. $\overleftrightarrow{DB}$ contains points A and C but $\overline{DB}$ contains neither point A nor point C. A belongs to $\overrightarrow{DB}$ but C does not.



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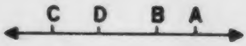


23. (a) 2
(b) 1
(c) 4
(d) 6
(e) 3
(f) 5
(g) 7
(h) 8

24. $x' = 8 - 6x$

25. (a) T
(b) T
(c) T
(d) T
(e) F
(f) T
(g) T
(h) T
(i) T
(j) F

26. (a) The coordinates of M and N are -2 and 6, respectively.
(b) -2, 6
(c) 2

27. (a) -4 (b) -5 (c) 9 (d) 17 (e) 5

28. (a) $x' = x_1 + k(x_2 - x_1)$
(b) $x' = -3 + 4k$
(1) $x' = 9$
(2) x' is any number such that $-3 \leq x' \leq 1$.
(3) $x' = -11$
(4) $x' = -1$

Part II

1. true
2. positive
3. $n \geq 0$
4. true
5. true
6. equal (or the same point)
7. PQ
8. informal
9. real number
10. origin (or unit-point)
11. $x - y$ if $x \geq y$
$y - x$ if $y > x$.
12. exactly two. Other answers which are acceptable but not as good are (1) at least one (2) at least two.
13. may be 0, 1 respectively; or may be any two real numbers
14. may
15. true
16. infinitely many
17. true
18. R bisects $\overline{PQ}$
19. real
20. may be 11; or must be 11 or -7.

Answers to Illustrative Test Items

Chapter 4

Part I

1. every
2. intersection
3. halfline
4. union
5. $\angle BAC$
6. true
7. true
8. ray-coordinates
9. true
10. positive number
11. true
12. true
13. true
14. be concurrent
15. interior
16. true
17. true
18. 180
19. true
20. congruent
21. true
22. true
23. $\overrightarrow{BA}$ and $\overrightarrow{BC}$
24. no
25. true

Part II

1. (e) none of these
2. (a) G
3. (d) vertical angles
4. (c) complementary angles
5. (a) a right angle
6. (c) They are both right angles.
7. (c) $180 - 2r$
8. (b) $\angle TAS$

Part III

1. (a) Acute
   (b) 120
   (c) 90
   (d) Perpendicular
   (e) Congruent
   (f) 30
   (g) Complementary
   (h) Obtuse
   (i) 90
   (j) Congruent (or acute)
   (k) Complement
   (l) Obtuse
   (m) Right
   (n) 120
   (o) 30

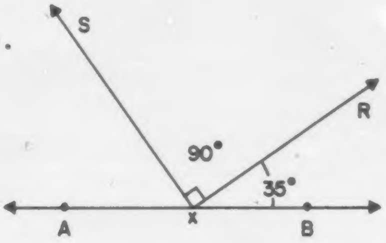
2. (a) {X, Z, W, Y, C}
   (b) $\emptyset$, that is, the empty set
   (c) {C}
   (d) {Z, W, X, Y}
   (e) Interior of polygon CXZWY

3. (a) x, $180 - x$, $180 - x$. (b) 180 (c) 20
   (d) Right (e) 179 (f) 41, 49

4. (a) 150 (b) 60 (c) $180 - n$ (d) $135 + n$

5. (a) 52 (b) 41 (c) $90 - n$ (d) $65 - n$

6. (a) $\overrightarrow{XR}$ and $\overrightarrow{XS}$.
   (b) $\angle BXR$ and $\angle SXA$.
   (c) None occur in the figure.
   (d) $\angle RXA$ and $\angle RXB$,
   $\angle BXS$ and $\angle AXS$.
   (e) $\angle SXA$ and $\angle RXB$.
   (f) $\angle SXB$ and $\angle RXA$.



7. (a) (1) $b - a$ (2) $\frac{c - b}{2}$
   (b) $p = \frac{c + b}{2}$

8. (a) Complements of congruent angles are congruent.
   (b) Supplements of congruent angles are congruent.
   (c) Vertical angles are congruent.

9. (a) Yes (b) No (c) (1) Yes
   (2) Not necessarily

10. (a) 100
    (b) 80
    (c) 140
    (d) 40
    (e) 100
    (f) 140

Answers to Illustrative Test Items

Chapter 5

A. 1. $\triangle ABW \cong \triangle MKF$

2. (a) QGWS
   (b) GHQ
   (c) HWS
   (d) GQHWS

3. (a) $\triangle FAH$
   (b) $\triangle FMR$
   (c) $\triangle MAH$

B. 1. (a) +
   (b) $m\angle A = m\angle D$, or $\angle A \cong \angle D$.
   (c) +
   (d) +
   (e) +
   (f) +
   (g) +
   (h) $\angle ACB \cong \angle DFE$, or $\angle ABC \cong \angle DEF$

2. (a) $\angle DAC$
   (b) $\angle CAB$, $\angle B$ (In either order)

3. (a) They have the same measure.
   (b) They have the same length.
   (c) Isosceles
   (d) The interior of $\angle XYZ$; $\angle XYS \cong \angle ZYS$
   (e) Median

4. Median; vertex angle

5. (a) Yes
   (b) No. We do not know $\overline{AB} \cong \overline{CD}$. Therefore we cannot use S.S.S. postulate.

   We do not know $\angle AEB \cong \angle CED$ and cannot use S.A.S. postulate.

   These angles are not vertical angles unless $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ are intersecting lines which would make points A, B, E, C, and D coplanar.

6. (a) A.S.A.
(b) I.E.
(c) S.A.S.
(d) S.S.S.
(e) S.A.S.
(f) I.E.
(g) S.A.S.
(h) I.E.
(i) I.E.
(j) S.A.S.
(k) S.A.S.
(l) S.S.S.
(m) S.A.S.
(n) S.A.S.

7. (a) S.A.S. or A.S.A.
(b) AD and CD, or ∠ABD and ∠CBD
(c) S.A.S.
(d) ∠U and ∠S, or VT and RT
(e) A.S.A.
(f) S.S.S.

8. (a) S.S.S. or S.A.S.
(b) S.A.S. or A.S.A.
(c) A.S.A.
(d) I.E.
(e) S.A.S.
(f) I.E.
(g) I.E.
(h) I.E.
(i) S.A.S. or A.S.A.
(j) A.S.A.
(k) A.S.A. or S.A.S.
(l) S.S.S. or S.A.S.
(m) A.S.A. or S.A.S.

9. (a) True
(b) False
(c) False
(d) False
(e) True

10. (a) No
(b) Yes
(c) Yes
(d) No

C. 1. (a) measure
(b) congruent, opposite, congruent
(c) congruent
(d) congruent, sides, congruence
(e) S.A.S. A.S.A. S.S.S.
(f) isosceles, equilateral
(g) one
(h) vertex
(i) median
(j) greater than

2. (a) (4)
   (b) (6)
   (c) (2)
   (d) (1)
   (e) (5)
   (f) (8)
   (g) (3)
   (h) (7)

3. (a) If angles have the same measure, they are congruent; and if angles are congruent, they have the same measure.
   (b) If two rays form a right angle, the lines determined by the rays are perpendicular; and if two lines are perpendicular, they contain two rays which form a right angle.

4. (a) If two angles are supplementary, they form a linear pair. False.
   (b) If two angles are right angles, they are congruent. True.
   (c) If $x = 5$, $x + 4 = 9$. True.
   (d) If two angles are congruent, they are vertical angles. False.
   (e) If y is a negative integer, then $y < 0$. True.
   (f) If two segments are congruent, they have the same length. True.

5. (a) True
   (b) False
   (c) True
   (d) True
   (e) False

6. (a) (3)
   (b) (2)
   (c) (5)
   (d) (6)
   (e) (6)
   (f) (4)
   (g) (5)
   (h) (1)

D. 1. 1. If two angles form a linear pair, they are supplementary.
      2. Hypothesis
      3. Supplements of congruent angles are congruent.
      4. Reflexive property of congruence
      5. A.S.A.
      6. Definition of congruence

2. Consider triangle ABC as isosceles with BC = AC, then $\angle A$ and $\angle B$ are base angles and $\angle A \cong \angle B$ by the isosceles triangle theorem. Now consider $\triangle ABC$ as isosceles with AC = AB, then $\angle B \cong \angle C$ by the isosceles triangle theorem. Therefore, $\angle A \cong \angle B \cong \angle C$ by the transitive property of congruence.

E. 1.

1. FA = FD	1. Hypothesis
2. $\angle A \cong \angle D$	2. Base angles of an isosceles triangle are congruent.
3. AB = DC	3. Hypothesis
4. $\triangle AFB \cong \triangle DFC$	4. S.A.S.
5. $\angle ABF \cong \angle DCF$	5. Definition of congruence for triangles
6. $\angle FBC \cong \angle FCB$	6. Supplements of congruent angles are congruent.

2.

1. $\overline{AH} \cong \overline{BF}$	1. Hypothesis
2. $\overline{AB} \cong \overline{BA}$	2. Reflexive property of congruence
3. $\angle r \cong \angle s$, and $\angle x \cong \angle y$	3. Hypothesis
4. $\angle HAB \cong \angle FBA$	4. Betweenness-Addition Theorem for angles
5. $\triangle HAB \cong \triangle FBA$	5. S.A.S.
6. HB = FA	6. Definition of congruence

3.

1. $\overline{AF} \cong \overline{BR}$	1. Hypothesis
2. $\overline{AB} \cong \overline{FR}$	2. Betweenness-Addition Theorem for segments
3. $\angle A \cong \angle R$	3. Hypothesis
4. $\angle x \cong \angle y$	4. Hypothesis
5. $\triangle BAN \cong \triangle FRH$	5. A.S.A.
6. $\overline{AN} \cong \overline{RH}$	6. Definition of congruence

4.

1. $\overline{AC} \cong \overline{BC}$	1. Hypothesis
2. $\angle A \cong \angle B$	2. Isosceles triangle theorem
3. $\overline{AE} \cong \overline{BH}$	3. Hypothesis
4. $\overline{PE} \perp \overline{AB}$ $\overline{QH} \perp \overline{AB}$	4. Hypothesis
5. $\angle PEA$ and $\angle QHB$ are right angles.	5. Perpendiculars determine right angles.
6. $\angle PEA \cong \angle QHB$	6. Right angles are congruent.
7. $\triangle PAE \cong \triangle QHB$	7. A.S.A.
8. $PE = QH$	8. Definition of congruence

5.

1. $AB = DE$	1. Hypothesis
2. $\overline{CM}$ and $\overline{FP}$ are medians.	2. Hypothesis
3. M and P are midpoints of $\overline{AB}$ and $\overline{DE}$ respectively.	3. Definition of median
4. $AM = DP$	4. Definition of midpoint and multiplication property of equality
5. $\overline{CM} \cong \overline{FP}$	5. Hypothesis
6. $AC = DF$	6. Hypothesis
7. $\triangle AMC \cong \triangle DPF$	7. S.S.S.
8. $\angle A \cong \angle D$	8. Definition of congruence
9. $\triangle ABC \cong \triangle DEF$	9. S.A.S.

Note: A proof in which the final reason is S.S.S. is also possible if $\triangle CMB$ is proved congruent to $\triangle FPE$.

6.

1. $\overline{AB} \cong \overline{CD}$, and $\overline{AD} \cong \overline{CB}$	1. Hypothesis
2. $\overline{BD} \cong \overline{DB}$	2. Reflexive property of congruence.
3. $\triangle ABD \cong \triangle CDB$	3. S.S.S.
4. $\angle EDF \cong \angle GBF$	4. Definition of congruence
5. $\overline{DF} \cong \overline{BF}$	5. Definition of bisects
6. $\angle EFD \cong \angle GFB$	6. Vertical angles are congruent.
7. $\triangle FDE \cong \triangle FBG$	7. A.S.A.
8. $EF = GF$	8. Definition of congruence

7.

1. $\overline{PE} \perp \overline{EP}$, and $\overline{PE} \perp \overline{EB}$	1. Hypothesis
2. $\angle PEA$ and $\angle PEB$ are right angles.	2. Perpendiculars determine right angles.
3. $\angle PEA \cong \angle PEB$	3. Right angles are congruent.
4. $\overline{PE} \cong \overline{PE}$	4. Reflexive property of congruence
5. $EA = EB$	5. Hypothesis
6. $\triangle PEA \cong \triangle PEB$	6. S.A.S.
7. $\overline{PA} \cong \overline{PB}$	7. Definition of congruence
8. $\angle PAB \cong \angle PBA$	8. Base angles of an isosceles triangle are congruent.

8.

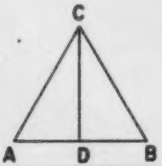
1. $\overline{AB} \cong \overline{FH}$	1. Hypothesis
2. $\overline{BH} \cong \overline{HB}$	2. Reflexive property of congruence
3. $\angle x \cong \angle g$	3. Hypothesis
4. $\triangle ABH \cong \triangle FHB$	4. S.A.S.
5. $\angle A \cong \angle F$	5. Definition of congruence
6. $\angle BHA \cong \angle HBF$	6. Definition of congruence
7. $\angle ABF \cong \angle FHA$	7. Betweenness-Angles Theorem and addition property of equality

9. (a) Yes (b) No; the Betweenness-Angles Theorem is true only if the rays are coplanar.

10.

1. CD = AB	1. Hypothesis
2. $\angle C$ and $\angle B$ are right angles.	2. Hypothesis
3. $\angle C \cong \angle B$	3. Right angles are congruent.
4. CE = BE	4. Definition of midpoint
5. $\triangle DCE \cong \triangle ABE$	5. S.A.S.
6. $\overline{DE} \cong \overline{AE}$	6. Definition of congruence

11. Hypothesis: $\triangle ABC$ is isosceles
with vertex at $\angle C$.
$\overline{CD}$ is a median.
Prove: $\overrightarrow{CD}$ bisects $\angle ACB$.



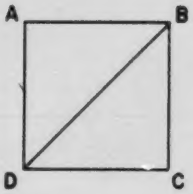
1. $\overline{AC} \cong \overline{BC}$	1. Definition of isosceles triangle
2. $\overline{AD} \cong \overline{DB}$	2. Definition of median
3. $\overline{CD} \cong \overline{CD}$	3. Reflexive property of congruence
4. $\triangle ACD \cong \triangle BCD$	4. S.S.S.
5. $\angle ACD \cong \angle BCD$	5. Definition of congruence
6. $\overrightarrow{CD}$ bisects $\angle ACB$.	6. Definition of angle bisector

(Another way of proving $\triangle ACD \cong \triangle BCD$ is to show $\angle A \cong \angle B$ and use S.A.S.)

12.

1. $\triangle ABC \cong \triangle WXY$	1. Hypothesis
2. $\angle CAB \cong \angle YWX$	2. Definition of congruence
3. $\angle DAB \cong \angle ZWX$	3. Step 2, definition of angle bisector and multiplication property of equality
4. $\overline{AB} \cong \overline{WX}$	4. Definition of congruence
5. $\angle B \cong \angle X$	5. Definition of congruence
6. $\triangle ABD \cong \triangle WXZ$	6. A.S.A.
7. $\overline{AD} \cong \overline{WZ}$	7. Definition of congruence

13. Hypothesis: ABCD is a quadrilateral with congruent sides and congruent angles, and diagonal $\overline{DB}$.



Prove: $\overline{DB}$ bisect ∠ADC and ∠ABC.

1. AB = BC AD = DC	1. Hypothesis
2. $\overline{DB} \cong \overline{DB}$	2. Reflexive property of congruence
3. Δ ABD ≅ Δ CBD	3. S.S.S.
4. ∠ABD ≅ ∠CBD ∠ADB = ∠CDB	4. Definition of congruence
5. $\overline{BD}$ bisects ∠ADC and ∠ABC	5. Definition of bisect

14.

1. ∠RTP ≅ ∠XPS	1. Hypothesis
2. $\overline{PT} \cong \overline{SP}$	2. Hypothesis
3. ∠PSO ≅ ∠TPO	3. Hypothesis
4. Δ RTP ≅ Δ XPS	4. A.S.A.
5. $\overline{RT} \cong \overline{XP}$	5. Definition of congruence

Answers to Illustrative Test Items

Chapter 6

I.

1. 0
2. +
3. +
4. 0
5. +
6. 0
7. 0
8. 0
9. +
10. +
11. 0
12. +
13. +
14. +
15. 0
16. +
17. 0
18. 0
19. +
20. +
21. +
22. +
23. 0
24. +
25. 0

II.

1. never
2. always
3. never
4. sometimes
5. sometimes
6. sometimes
7. always
8. always
9. always
10. sometimes
11. always
12. never

III.

1. 78 ; 102
2. $m\angle a = 108$; $m\angle b = 72$
3. 100 ; 80
4. $m\angle b = 52$
5. No
6. 70 (Consider a line through Y parallel to p and q .)
7. 140 (Consider a line through Y parallel to p and q .)
8. 50
9. (a) (1) 35 ; (3) 85 ; (5) 25 .
   (2) 70 ; (4) 155 ;
   (b) (1) False; (3) True.
   (2) True;
10. (a) (1) $m\angle a = 113$
    (2) $m\angle b = 43$
    (b) $AD > BD > AB$

11. (a) $\overleftrightarrow{CD} \parallel \overleftrightarrow{FE}$
(b) None
(c) $\overleftrightarrow{BG} \parallel \overleftrightarrow{DH}$
(d) $\overleftrightarrow{CF} \parallel \overleftrightarrow{DH}$
(e) None
(f) $\overleftrightarrow{AF} \parallel \overleftrightarrow{BG}$
(g) $\overleftrightarrow{AB} \parallel \overleftrightarrow{FE}$
(h) $\overleftrightarrow{AC} \parallel \overleftrightarrow{DH}$

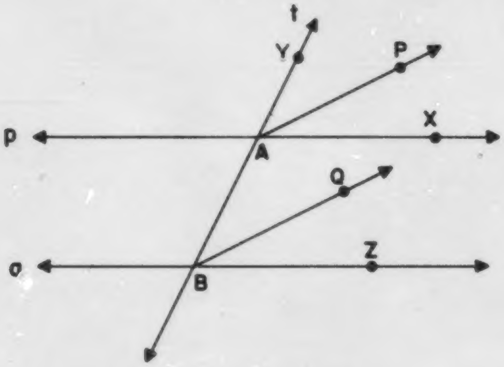
IV.

1. b
2. c
3. c
4. b
5. c

V.

1. Proof:

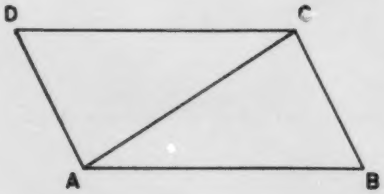
It is given that $p \parallel q$; t intersects p and q at A and B , respectively so that $\angle XAY$ and $\angle ZBA$ are a pair of corresponding angles; $\overrightarrow{AP}$ is the bisector of $\angle XAY$ and $\overrightarrow{BQ}$ is the bisector of $\angle ZBA$. We are required to prove $\overleftrightarrow{AP} \parallel \overleftrightarrow{BQ}$.



Statements	Reasons
1. $p \parallel q$.	1. Hypothesis.
2. $\angle XAY \cong \angle ZBA$.	2. Corollary 6-4-1.
3. $\overrightarrow{AP}$ bisects $\angle XAY$, $\overrightarrow{BQ}$ bisects $\angle ZBA$.	3. Hypothesis.
4. $m\angle PAY = \frac{1}{2} m\angle XAY$, $m\angle QBA = \frac{1}{2} m\angle ZBA$.	4. Definition of bisect.
5. $m\angle PAY = m\angle QBA$.	5. Multiplication property of equality.
6. $\overleftrightarrow{AP} \parallel \overleftrightarrow{BQ}$.	6. Corollary 6-2-1.

2. Proof:

It is given that $AB = CD$ and $AD = BC$. We are required to prove $\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \parallel \overline{BC}$.



$\triangle CAB \cong \triangle ACD$ by S.S.S. Hence, $\angle CAB \cong \angle ACD$ and $\angle ACB \cong \angle CAD$. Therefore, $\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \parallel \overline{CB}$ by Theorem 6-2.

3. Proof:

It is given that $x \parallel y$ and $p \parallel q$. We are required to prove $\angle a \cong \angle b$. Let $\angle c$ be the angle formed by lines y and q, such that the interior of $\angle c$ intersects both the interior of $\angle a$ and the interior of $\angle b$. Then, $\angle a$ is a supplement of $\angle c$ and $\angle b$ is a supplement of $\angle c$. (Corollary 6-4-2) Hence, $\angle a \cong \angle b$. (Supplements of congruent angles are congruent.)

4. Proof:

Statements	Reasons
1. $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.	1. Hypothesis.
2. $\angle DCB \cong \angle ABC$.	2. Theorem 6-4.
3. $AB = AC$.	3. Hypothesis.
4. $\angle ABC \cong \angle ACB$.	4. Theorem 5-6.
5. $\angle DCB \cong \angle ACB$.	5. Transitive property of congruence of angles
6. $\overrightarrow{CB}$ is a midray of $\angle DCA$.	6. Definition of a midray.

5. Proof:

Statements	Reasons
1. $\overline{AB} \parallel \overline{CD}$.	1. Hypothesis.
2. $\angle ABD \cong \angle CDB$.	2. Theorem 6-4.
3. $\angle y \cong \angle x$.	3. Hypothesis.
4. $\angle A \cong \angle C$.	4. Theorem 6-11.

(or prove $\triangle ABD \cong \triangle CDB$ by A.S.A and $\angle A \cong \angle C$ by definition of congruence for triangles.)

6. The Parallel Postulate proves the problem.

7. Proof:

Statements	Reasons
1. $AB = BC$.	1. Hypothesis.
2. $m \angle C = m \angle A$.	2. Theorem 5-6.
3. $m \angle ADB > m \angle C$.	3. Theorem 5-10 .
4. $m \angle ADB > m \angle A$.	4. Substitution property of equality.
5. $AB > BD$.	5. Theorem 6-18.

8. Proof:

Statements	Reasons
1. $\overrightarrow{TV}$ bisects $\angle RTW$.	1. Hypothesis.
2. $\angle RTV \cong \angle WTV$.	2. Definition of bisect.
3. $\overleftrightarrow{TV} \parallel \overleftrightarrow{SW}$.	3. Hypothesis.
4. $\angle WST \cong \angle RTV$.	4. Corollary 6-4-1.
5. $\angle WTV \cong \angle SWT$.	5. Theorem 6-4.
6. $\angle WST \cong \angle SWT$.	6. Transitive property of congruence of angles.
7. $\overline{WT} \cong \overline{ST}$.	7. Theorem 5-7.
8. $\triangle STW$ is isosceles.	8. Definition of isosceles triangle.

Answers to Illustrative Test Items

Chapter 7

1. (a) 33 .
   (b) 9 .
   (c) $\sqrt{2}$.
   (d) 0 .
   (e) 0 .
   (f) $\frac{12}{\sqrt{3}}$ or $4\sqrt{3}$.
   (g) -2.8 .
2. $x = 4$, $y = 3$, $z = 4$.
3. (1) f and h are similar, a and e are similar.
   (2) b , d .
4. Transitive property of similarity.
   12 .
5. $DB = 3$, $AC = 2$, $BC = 2\sqrt{3}$, $AB = 4$.
6. 45 .
7. $\frac{60}{\sqrt{3}}$ or $20\sqrt{3}$.
8. $3\sqrt{3}$.
9. 16 .
10. 8 .
11. 18 .
12. $\overleftrightarrow{MN}$ intersects $\overleftrightarrow{KL}$.
13. $\triangle AEB \sim \triangle CED$ by S.A.S. Similarity Theorem.
    Therefore $\angle B \cong \angle D$.

14. If $\overline{BD} \parallel \overline{CE}$, $\angle ABD \cong \angle ACE$ since they are corresponding angles formed when the parallel lines are cut by transversal $\overleftrightarrow{AC}$. $\angle CAE \cong \angle BAD$ by the Reflexive Property of Congruence. $\triangle ABD \sim \triangle ACE$ by the A.A. Similarity Theorem. Thus $AB = k \cdot AC$, $BD = k \cdot CE$, since these are the corresponding sides which are proportional. Since B is the midpoint of $\overline{AC}$, $AB = \frac{1}{2}AC$ by the definition of midpoint. Thus the proportionality constant $k = \frac{1}{2}$ and, substituting, we see that $BD = \frac{1}{2}CE$.

15. (a) Not necessarily true.
    (b) True. By alternation.
    (c) True. By Definition of Proportionality.
    (d) Not necessarily true.
    (e) True. By the Products Property.
    (f) True. By Inversion.

16. See the statement of the S.A.S. Similarity Theorem.

Chapter 1

ANSWERS AND SOLUTIONS

Problem Set 1-4

Problem 5 provides drill in inductive reasoning. Problem 5 is intended to be less obvious than Problem 4.

Problems 6-8 point up some pitfalls in inductive reasoning.

Problem 9 pursues the exploration begun in the text. It is doubtful that a sophomore or junior can present the requested proof without peeking ahead to page 12 where a proof is given. The intent of the problem is to illustrate how deductive reasoning can sometimes be applied to resolve a question which inductive reasoning cannot completely resolve.

1. (a) Angle a and angle b appear to have the same measure. Angle c and angle d appear to have the same measure.
   (b) Angle a and angle b appear to contain the same number of degrees. This appears to be also true for angles c and d.
   (c) Yes. If two lines intersect, they form four angles and the angles "opposite each other" have the same measure.
   [If the student knows what vertical angles are, he may use that term.]
2. (a) Angles opposite the sides having equal lengths appear to have the same "size".
   (b) Same as (a).
   (c) If two sides of a triangle have equal lengths, then the angles opposite these sides have equal measures.
3. The total number of degrees in the three angles of a triangle is 180.
4. The total number of degrees in the four angles of a quadrilateral is 360. Measure the angles in more quadrilaterals.
5. Every product is approximately 900.

Generalization: If two chords of a circle intersect, the product of the lengths of the "segments" of one is the same as the product of the lengths of the "segments" of the other.

[Students should also note that the unit used does not affect this relation.]

6.

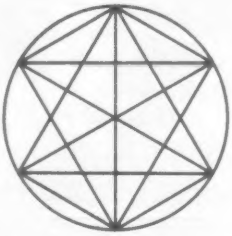
n	$n^2 - 2n + 2$
1	$1 - 2 + 2 = 1 = n$
2	$4 - 4 + 2 = 2 = n$
3	$9 - 6 + 2 = 5 \neq n$

$n^2 - 2n + 2 = n$ is not always true.

7. (b) or (d) or both.

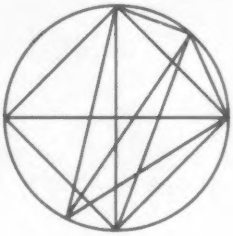
8. Students may plausibly predict that the pattern established for the number of regions will continue and give, for 6 points, $2^6 - 1$ or 32 regions. They may be reluctant to believe that the number of regions is:

30 for the arrangement in (a) where 3 diagonals are concurrent



or

31 for the arrangement in (b) where no 3 diagonals are concurrent.



9. $Q = n^2 + n + 11$; n represents any positive integer.

$Q = n(n + 1) + 11$

n and $(n + 1)$ are consecutive positive integers, hence, for all integral values of n, either n or $(n + 1)$ is an even number. $n(n + 1)$ has an even number as a factor and is, therefore, an even number. The sum of an even number and an odd number is always odd. Therefore, $Q = n(n + 1) + 11$ is an odd number if n is any positive integer. (Also, see discussion on page 12.)

Problem Set 1-5

All problems of this set are intended to be drill on the meaning of "deductive reasoning", and its contrast with "inductive reasoning". All problems are suitable for homework assignments, although the solutions are not obvious in every case. Sight reading is dangerous here, even for teachers.

Note particularly problem 1(i) and 1(j), not because they are difficult, but because they illustrate the unreliability of conclusions deduced from false hypothesis.

1. (a) Miss Smith is John's teacher unless John is a visitor, a stray dog--not a student.
   (b) My father has climbed a mountain.
   (c) Since the first statement gives a prerequisite for all policemen in the city, it could have been written "all policemen in Elk City are at least 6 ft. tall."

   Conclusion: Jim's uncle is at least 6 feet tall.

   (d) Harry has passed test R.
   (e) No conclusion. The statement did not indicate that going to the beach on Saturday was restricted to seniors. Alice may not have been a senior.
   (f) No conclusion. Though we infer from the first statement that all children under 12 years of age ride on buses for half fare, the statement does not exclude the possibility that other groups of people might also be given that privilege. Jack may belong to another privileged group.

(g) No conclusion. If the second statement had said "A fly is an insect.", then the conclusion that the fly has six legs would follow. Also, if the first statement had been "Only insects have six legs.", then the fly (with six legs) must be an insect.

(h) The mathematical interpretation would include an "all" before "Rainy days." Thus, the conclusion is "Friday was a disagreeable day."
Common usage might result in understanding the first statement to mean "Rainy days are generally disagreeable," or "Some rainy days are disagreeable."

This problem will probably lead to a discussion concerning the difference between a precise statement and one of which the meaning is open to question.

(i) and (j) are examples in which the general hypothesis is not true since there are varieties of apples which are not red when ripe and many trees which do not have needles. In both (i) and (j) the second statement is true. In (i) the logical conclusion is: "Early transparent apples are red when ripe." This is false since the early transparent is yellow when ripe. In (j) the logical conclusion is: "Fir trees have needles." The conclusion is true. These two problems offer opportunity to point out that deductive reasoning may or may not lead to a conclusion we can label "true." If the hypotheses are all true and if the thinking which led to the conclusion is deductive thinking, then the conclusion will be true. All of this will, of course, have more meaning for the student later in the course than at the introductory stage of his study.

2. (a) There are 12 months to be paired with 25 students. No month need be repeated in listing birthdays of the first 12 students; but the birthday of the 13th student must repeat a month already named, if there was no earlier repetition.

(b) Let t represent the number of trees and let ℓ be the greatest of all those numbers which actually give the number of leaves on some tree. If a list is drawn up pairing each tree with the number of leaves on it, then no two trees of the first ℓ on this list need have the same number of leaves. However, if there is no duplication among the first ℓ then the next tree on the list must yield a duplication. (The assumption that t is greater than ℓ guarantees that there must be such a next one.)

3. 32 dominoes are required. Each domino placed in *any* horizontal or vertical position covers one white and one colored square. No single domino can cover *two* white squares. Removing the two black squares leaves two unpaired white squares, each requiring a different domino to cover it.

4. (a) Possible conclusion: This part of the country never has snowstorms. An induction.

   (b) One of several possible conclusions: I cannot get a suit for $25 at that store. A deduction.

   (c) Possible conclusion: All cells of that type have a thick cell wall. An induction.

   (d) Conclusion: The medical center in Springdale is a steel and glass type structure. A deduction.

   (e) Conclusion: Jerry has passing grades. A deduction.

Chapter 2

ANSWERS AND SOLUTIONS

Problem Set 2-1

All of the problems in this set, except perhaps some parts of Problem 8, should be considered by all students for whom this is the first experience with sets.

1. belongs; member; contains.

2. (c) {2, 4, 6, 8, 10, 12, 14}
   (d) {1, 4, 9, 16, 25, 36, 49}

3. (a) {1, 4, 9}
   (b) No. The element, 1, which is in the first set is not contained in the set {4, 9}.

4. the set of odd numbers from 3 to 9 inclusive;
   the set of odd numbers from 2 to 10;
   the four consecutive odd numbers larger than 2.

5. (a) and (b)

6. The same. The equality of sets is independent of the arrangement of the elements in each set.

7. (a) 9; -9; {9, -9}
   (b) {4, -4}

8. (a) {3}
   (b) $\{\frac{5}{2}\}$ or {2.5}
   (c) $\{\frac{18}{5}\}$ or $\{3\frac{3}{5}\}$
   (d) {5, -5}
   (e) {5, -2}

9. (a) Complete list possible. {10, 12, 14, 16, ...}
   (b) possible; {1, 2, 3, 4, ...}
   (c) impossible; $\{\frac{51}{50}, \frac{5}{4}, \frac{4}{3}, \frac{3}{2}, \ldots\}$
   (d) impossible; {0, -1, +1, -2, ...}
   (e) impossible; {0, -7, +7, -14, ...}
   (f) possible; {1, 2, 3, 5, 6, 10, 15, 30}
   (g) possible; {2, 3, 5}

10. (a) $\sqrt{2}$, $-\frac{1}{3}$
    (b) 6, -81, 0, 108.29, 108.28
    (c) 6, $\sqrt{2}$, $-\frac{1}{3}$, -81, 0, 108.28, 108.29.

Problem Set 2-2a

Students who have had no previous experience with sets should do all of these. The idea of "proper subset" brought out in Problem 4 is not used in the further development in this text.

1. (a) Union
   (b) Intersection
   (c) Contained

2. Union

3. {N}

4. {2, 4, 6} ; {2} ; {4} ; {6} ; {2, 4} ; {2, 6} , {4, 6} , { }
   all are proper subsets except {2, 4, 6}.

   (Students are not expected to include the last subset. The empty set or null set is discussed in the next section.)

5. One of many possible answers.
   (5, 7, -3) (5, 9, x) (7, 9, y) (-3, x, y)

Problem Set 2-2b

1. {5, 9, 11} ; {3, 4, 5, 6, 7, 9, 10, 11, 12}

2. (a) S_1 and S_2; S_1 and S_3; S_1 and S_5; S_2 and S_5 if you are a boy; S_3 and S_5 if you are a girl.
   (b) S_1
   (c) S_1
   (d) The set consisting of all members of faculty and students of your school.
   (e) S_1, S_2, S_3, S_5

3. (a) Sets L and M and sets M and N intersect.
   (b) The intersection of sets L and M is a set with one element: {t}.
   The intersection of sets L and N is the empty set.

4. (a) Case I: Point A; Case II: Points B and C; Case III: The empty set.
   (b) The intersection would be a set of all points on the lines between B and C, and also Points B and C.
   (c) Answers would not change.

5. (a) b, c
   (b) The set {J}
   (c) The set {K}
   (d) The same point.
   (e) The empty set.

6. (a) The set of all positive integers divisible by 6.
   {6, 12, 18, 24, ...}
   (b) The set of all positive integers divisible by either 2 or 3.
   {2, 3, 4, 6, 8, 9, 10, 12, ...}

7. One way of solving this problem is by actually listing all of the members of each set and picking out the ones that are common to all.

A better approach would be to choose the numbers common to two subsets and then get the intersection of those elements with the other subsets. If we consider $(b) \cap (c)$, we have all the two-digit numbers that contain a 7 and whose digit sum is even. The latter statement means that both digits must be odd, or
$(b) \cap (c) = \{17, 37, 57, 71, 73, 75, 77, 79, 97\}$.
Now consider $[(b) \cap (c)] \cap (d)$. This means we eliminate all those that have the unit's digit less than or equal to the ten's digit, or
$\{17, 37, 57, \cancel{71}, \cancel{73}, \cancel{75}, \cancel{77}, 79, \cancel{97}\}$.

Now to get the final intersection, we simple choose the prime numbers or
$\{17, 37, 79\}$.

This problem will be useful to illustrate to the class that there are frequently many approaches that attain the same result. However, some of them are more economical than others. It will be profitable to have various students describe their procedures.

8. (a) the intersection of M and N; the union of M and N.
   (b) $M \cap N = \{a, e\}$; $M \cup N = \{a, b, c, d, e, f, g\}$

9. (a) The empty set
   The set of all males
   (b) $A \cup B = \{1, 2, 3, 4, \dots\}$
   $A \cap B = \{\ \ \}$ or $\emptyset$

10. (a) $\{3, 6, 7, 9, 12, 14, 15, 18, 21, 28\}$
    (b) $\{\ \ \}$ or $\emptyset$
    (c) $\{6, 12\}$
    (d) $\{6, 12\}$

Problem Set 2-3

1. (a) $A \leftrightarrow Z$
$B \leftrightarrow X$
$D \leftrightarrow Y$

(b) $A \leftrightarrow Y$
$B \leftrightarrow X$
$D \leftrightarrow Z$

2. (a) $A \leftrightarrow U$
$B \leftrightarrow V$
$D \leftrightarrow X$
$E \leftrightarrow Y$
$G \leftrightarrow W$

(b) $A \leftrightarrow V$
$B \leftrightarrow U$
$D \leftrightarrow X$
$E \leftrightarrow Y$
$G \leftrightarrow W$

(c) $A \leftrightarrow X$
$B \leftrightarrow U$
$D \leftrightarrow V$
$E \leftrightarrow Y$
$G \leftrightarrow W$

3. No. If two sets do not have the same number of elements, then there can be no one-to-one correspondence between them.

4. (a) 7 (b) $-\sqrt{2}$ (c) $-(\sqrt{5}-2)$ (d) Yes (e) $-p$
(f) No. If n is an element of N, $-n$ is its unique matching element in P (g) No. If p is an element of P, $-p$ is its unique element in N.
(h) Yes

5. (a) 4.3 (b) 3.3 (c) $s = t - 2$ (d) Yes
(e) Yes (f) Yes (g) Yes

6. A one-to-one correspondence exists between two sets U and W if each element of U is matched with exactly one element of W and each element of W is matched with exactly one element of U. (Every element in each set is used exactly once in the matching.) In the given correspondence $S \longleftrightarrow S$, where each number was matched with itself, these conditions are satisfied.

Problem Set 2-4

1. Statement (1) follows logically from statements (a) and (c).

2. Statement (2) follows logically from statements (b) and (d).

3. There may be more than 3 pins.

4. There may be more than 2 lins.

5. Statement (5) follows logically from statements (a), (d), and (c).

6. Statement (c) implies that if 2 lins contain the same pair of distinct pins the 2 lins are one and the same, not distinct. Therefore, two distinct lins cannot have more than one pin in common.

Problem Set 2-5

1. {A, B, C}

2. (a) False. The points may not lie in the same line.
   (b) Ture, because of Postulate 2.
   (c) False. "At least" leaves a possibility of any number beyond two, all of which need not lie on the same line.

3. (a) {A, B, E}, {A, C, D}, {B, C, F}.
   (b) {A, B, E, C}; {A, B, E, D}; {A, B, E, F}.
   {A, C, D, B}; {A, C, D, E}; {A, C, D, F}.
   {B, C, F, A}; {B, C, F, D}; {B, C, F, E}.
   (c) {E, B, C, D}; {D, E, C, F}; {A, B, D, F}.
   {B, E, F, D}; {A, E, C, F}; {A, E, F, D}.
   (d) $\overleftrightarrow{AF}$, $\overleftrightarrow{BD}$, $\overleftrightarrow{CE}$, $\overleftrightarrow{DE}$, $\overleftrightarrow{DF}$, $\overleftrightarrow{EF}$

4. By Theorem 2-2, space contains at least three distinct points not in one line. Call these points A, B, and C. Then, by Postulate 3, A and B determine a line, B and C determine a line, A and C determine a line. Then space contains lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, and $\overleftrightarrow{AC}$ and thus contains at least three lines.

5. Postulate 1 and 3. We are given one point, say A. Postulate 1 says that space contains at least two distinct points. Thus, there is another point besides A, say B, in space. Postulate 3 says that given two distinct points, there is one and only one line that contains those points. Thus, there is a line containing A.

Problem Set 2-6

Problems 1, 2, and 3 of this set are easy enough. Problem 4 involves indirect proof and may stump all but the best students at this stage. Problem 5 deals with an intuitively strange situation but, by itself, might make an intriguing homework assignment.

1. Postulate 6. The points of contact of the legs of a tripod with the surface upon which they rest determine a plane. The points of contact of the legs of a four legged table with the surface upon which it rests may or may not be in the same plane. Four, noncoplanar, noncollinear points determine four planes. Shifting from one to another of these four planes would have the effect of rocking.

2. $F = G$ by Postulate 6.

3. (a) Yes. For example, consider a set consisting of two points, P and Q. This set is in the line $\overleftrightarrow{PQ}$. It also is in the plane PQR where R is a point not on $\overleftrightarrow{PQ}$ (such points exist by Postulate 4). Another example would be line $\overleftrightarrow{PQ}$ itself not just the points P and Q.

   (b) Yes. Let ℓ be the line that contains the collinear points. By Postulate 4, there is a point R not on ℓ. By Postulate 2, there are two points A and B in ℓ. The points A, B, and R are noncollinear and determine, by Postulate 6, a unique plane, $\mathcal{M}$. Since points A and B of ℓ are in $\mathcal{M}$; by Postulate 8, ℓ is contained in $\mathcal{M}$. Thus, the collinear points are contained in $\mathcal{M}$, and are coplanar.

4. (a) Let A, B, C, D be the 4 distinct noncoplanar points. Either they are collinear or not collinear. Assume they are collinear. Then, as in problem 3(b) above, they are coplanar. But this contradicts the hypothesis that the points are noncoplanar. We must, therefore, discard the assumption that A, B, C, D are collinear and accept the conclusion that they are not collinear.

(b) Call the 4 distinct, noncoplanar points A, B, C, D. Either 3 points, say A, B, C, are collinear or they are not collinear. Assume they are collinear. Then by 4(a) D can not be collinear with A, B, C. By Postulate 6, there is one and only one plane containing A, B, and D. But, by Postulate 8, C which is on $\overleftrightarrow{AB}$ also lies in that plane. All points would then be coplanar. This contradicts the hypothesis and we must discard the assumption that A, B, C are collinear. A, B, C must, therefore, be noncollinear.

5. (a) No, since that would contradict Postulate 7.

(b) No, since by the argument of Problem 4(a) above, 4 distinct noncoplanar points are noncollinear.

(c) The plane contains exactly 3 points. Since, by Postulate 5, every plane must have at least 3 points and, since all points cannot be coplanar, the 4th point cannot lie in the same plane as the other three.

(d) No three of the four points may be collinear by the argument of 4(b) above.

(e) A line contains exactly 2 points. A line must contain at least 2 points. By the argument of part (d), a line cannot contain 3 points. By the argument of part (b), a line cannot contain 4 points. It must then contain exactly 2 points.

(f) $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{AD}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{BD}$, $\overleftrightarrow{CD}$.

(g) Plane ABC, plane ABD, plane ACD, plane BCD.

Chapter 2

Review Problems

1. (a) $S = \{ * \}$
   (b) $S = \{ *, \neq, \# \}$
   (c) $S = \{ \quad \}$ or $\emptyset$
   (d) $S = \{ *, -, +, \div, \times \}$
   (e) $S = \emptyset$
   (f) $S = \{ *, \neq, \#, -, +, \div, \times \}$

2. These points determine one line.

3. (a) One line contains all of them
   (b) (1) A (3) E
   (2) C (4) noncollinear, or coplanar

4. (a) One plane contains all of them.
   (b) (1) P, Q, M (3) Q, T, R, M
   (2) Q, T, R, P (4) R, Q, T

5. (a) $2 \longleftrightarrow 4$
   $3 \longleftrightarrow 9$
   $4 \longleftrightarrow 16$

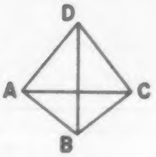
   The correspondence is unique.

   (b)
$$\begin{array}{llll} 4 \longleftrightarrow 3; & 4 \longleftrightarrow 3; & 4 \longleftrightarrow 2; & 4 \longleftrightarrow 2 \\ 9 \longleftrightarrow 2; & 9 \longleftrightarrow 4; & 9 \longleftrightarrow 3; & 9 \longleftrightarrow 4 \\ 16 \longleftrightarrow 4; & 16 \longleftrightarrow 2; & 16 \longleftrightarrow 4; & 16 \longleftrightarrow 3 \end{array}$$

   Four such correspondences are possible.

6. (a) Yes (b) Yes (c) No

7. (a) With our present definitions, postulates, and theorems, it appears that at least 3 different lines contain one given point.



(b) Exactly one line.
[The tetrahedron with its 4 noncoplanar points as vertices helps in visualizing the situation in Problems 7 and 8.]

8. (a) It appears that at least 3 planes may contain one given point.
(b) One plane, if the three points are noncollinear.
At least two planes, if the three points are collinear.
(c) At least two planes.

9. (1) b
(2) d
(3) c
(4) d
(5) a
(6) a

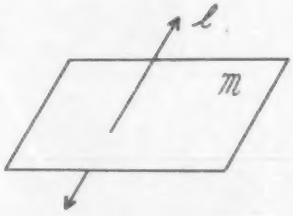
Yes, the labels are shorter and easier to locate on the diagram.

10. The conclusion is not certain. Assume A, B, C are distinct. A, B, C may be collinear, in which case, by Postulate 9, they may be on the intersection of two distinct planes $\mathcal{M}$ and $\mathcal{N}$.
If A, B, C are noncollinear, then $\mathcal{M}$ and $\mathcal{N}$ are the same plane by Postulate 6.

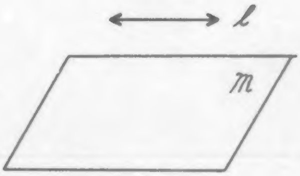
11. (1) three noncollinear points
(2) two intersecting lines
(3) a line and a point not on the line

12. No. Every point in ℓ_2 lies in $\mathcal{E}$. If ℓ_1 intersects ℓ_2 in some point M, then ℓ_1 contains two points P and M, both in $\mathcal{E}$. But this is impossible, since if a line intersects a plane which does not contain it, then the line and the plane have exactly one point in common.

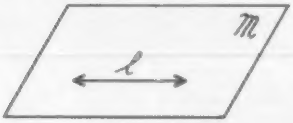
13. (a)



(b)

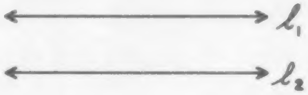


(c)

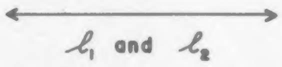


14. Yes. Postulate 9 says that if two planes intersect, then their intersection is a line. This means $\overleftrightarrow{AB}$ contains every point which lies in both planes $\mathcal{M}$ and $\mathcal{N}$ and must, therefore, contain P.

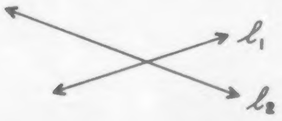
15. (a)



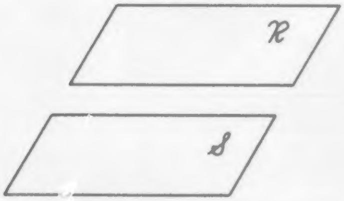
(c)


ℓ_1 and ℓ_2

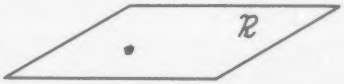
(b)



16. (a)

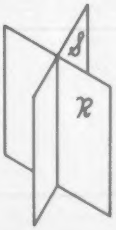

$\mathcal{R}$
$\mathcal{S}$

(c)



$\mathcal{R}$ and $\mathcal{S}$ are the same plane

(b)


$\mathcal{S}$
$\mathcal{R}$

270

17. (a) If zebras have polka dots, then they are dangerous.
    (b) If the sides of a rectangle have equal lengths, then the rectangle is a square.
    (c) If Oklahoma wins, then there will be a celebration.
    (d) If two lines intersect, then a plane is determined.
    (e) If a dog is a cocker spaniel, then it is sweet tempered.
    (f) If ℓ_1 and ℓ_2 are two distinct lines, then ℓ_1 and ℓ_2 have at most one point in common.
    (g) If a student takes geometry, then he knows how to add integers.
    (h) If ℓ is any line and P any point not on ℓ, then there is exactly one plane which contains both P and ℓ.

18. The single line indicates the hypothesis and the double line the conclusion:
    (a) If $\underline{\text{John is ill}}$, $\underline{\underline{\text{he should see a doctor}}}$.
    (b) If $\underline{\text{a person has red hair}}$, $\underline{\underline{\text{he is nice to know}}}$.
    (c) $\underline{\underline{\text{Four points are collinear}}}$ $\underline{\text{if they lie on one line}}$.
    (d) If $\underline{\text{I do my homework well}}$, $\underline{\underline{\text{I will get a good grade}}}$.
    (e) If $\underline{\text{a set of points lies in one plane}}$, $\underline{\underline{\text{the points are coplanar}}}$.

Chapter 3

ANSWERS AND SOLUTIONS

Problem Set 3-2

Problems 5 through 9 prepare for the formal statements of the properties of order. Thus Problems 5 and 6 illustrate the additive property of order and show that it includes subtraction. Problems 7 and 8 illustrate the multiplicative property using both positive and negative multipliers. Fraction multipliers point toward the fact that the multiplicative property includes division.

For Problem 6, suggest that the student get an idea of the basic relation with which he is working by letting $x = 0$ in each case before he replaces x by either a positive number or a negative number. Similarly, in problem 8, suggest that x be replaced by 1 before other replacements are used. It is hoped that students will decide from Problem 6 that adding either a positive or negative number to both sides of the inequality does not alter the truth value of the statement and from Problem 8 that, whereas multiplying both sides of an inequality by a positive number does not alter the truth value, multiplying by a negative number does alter the truth value.

1. (a) $E = F$
   (b) A, B, C, D, E, F are subsets of A.
       B, E, F are subsets of B.
       C, B, E, F are subsets of C.
   (c) B, (d) A, (e) B.

2. (a) $2 > -5$; 2 is to the right.
   (b) $-3 > -7$; -3 is to the right.
   (c) $5 > 3$; 5 is to the right.
   (d) $0 > -\frac{3}{2}$; 0 is to the right.
   (e) They are the same number and are associated with the same point on the number line.
   (f) $\frac{16}{3} > \frac{21}{4}$; $\frac{16}{3}$ is to the right.
   (g) $-\sqrt{\frac{25}{16}} > -\sqrt{\frac{16}{9}}$; $-\sqrt{\frac{25}{16}}$ is to the right.

3. It is not true.
For all integers a, b, c, if $a = b + c$ and if $c > 0$, then $a > b$.
It is useful to consider $a - b$. For a to be $> b$, $a - b$ must be > 0. But from the hypothesis we know that $a - b = c$. Therefore c must be > 0. This indicates that we must add to the original hypothesis the hypothesis $c > 0$ as in the modified form above.

4. (b) $2, -4 > -6$
(c) $-10, -2 > -12$
(d) $11, 8 > -3$
(e) $-6, 2 > -4$
(f) $\frac{1}{6}, \frac{1}{2} > \frac{1}{3}$
(g) $-6, 3 > -3$
(h) $a - b$ will be positive
$a > b$

*5. (a) $1 > -3$
(b) $1 + 5 > -3 + 5$
(c) $1 - 5 > -3 - 5$
(d) $-2 > -6$
(e) $-2 + 3 > -6 + 3$
(f) $-2 - 3 > -6 - 3$

*6. (a) (1) True (2) True
(b) (1) True (2) True
(c) (1) False (2) False
(d) (1) True (2) True
(e) (1) True (2) True
(f) (1) False (2) False

*7. (a) (i) $5 > 2$
(ii) $3 \cdot 5 > 3 \cdot 2$
(iii) $(-3) \cdot 2 > (-3) \cdot 5$
(b) (i) $2 > -6$
(ii) $3 \cdot 2 > 3(-6)$
(iii) $(-3) \cdot (-6) > (-3) \cdot 2$
(c) (i) $12 > 8$
(ii) $\frac{1}{2} \cdot 12 > \frac{1}{2} \cdot 8$
(iii) $(-\frac{1}{2}) \cdot 8 > (-\frac{1}{2}) \cdot 12$

*8. (a) (1) True (2) False (d) (1) False (2) True
(b) (1) False (2) True (e) (1) True (2) False
(c) (1) False (2) True (f) (1) False (2) True

*9. (a) (1) True (2) False
(b) (1) True (2) False
(c) (1) True (2) False

Problem Set 3-3a

In applying the properties of order, you may prefer the complete form of solution as used for Problem 3 for all of the problems.

1. (a) Additive property
   (b) Additive property
   (c) Transitive property
   (d) Transitive property. $a > 4$, $4 > 0$, then $a > 0$
   (e) Transitive property used twice
   (f) Additive property
   (g) Additive property
   (h) Multiplicative property using a negative multiplier
   (i) Additive property followed by the multiplicative property using a positive multiplier
   (j) Multiplicative property using a positive multiplier followed by the additive property

2. $x > y$, $y > z$ by hypothesis.
   Then $x > z$ by the transitive property of order.
   But $z > w$ by hypothesis. Therefore
   $x > w$ by the transitive property of order.

3. $2x - 3$ is positive by hypothesis.
   $2x - 3 > 0$ by definition of positive.
   $2x - 3 + 3 > 0 + 3$ by the additive property of order.
   $2x > 3$ by simplifying names.
   $\frac{1}{2} \cdot 2x > \frac{1}{2} \cdot 3$ by the multiplicative property of order using a positive multiplier.
   $x > \frac{3}{2}$ by simplifying names.

4.	$3 - 2x$ is positive	by hypothesis.
	$3 - 2x > 0$	by definition of positive.
	$3 > 2x$	by the additive property of order and simplifying names.
	$\frac{3}{2} > x$	by the multiplication property of order using a positive multiplicative and simplifying names.
5.	$5 - 10x$ is negative	by hypothesis.
	$0 > 5 - 10x$	by definition of negative.
	$10x > 5$	by additive property of order and simplifying names.
	$x > \frac{1}{2}$	by multiplicative property of order, using a positive multiplier and simplifying names.
6.	$y > 3$	by hypothesis.
	$0 > 3 - y$	by the additive property of order.
	$5 > 0$	by the definition of positive number.
	$5 > 3 - y$	by the transitive property of order.

Problem Set 3-3b

Problem 5(b) offers an appropriate place to call attention to the difference between justifying a "No" answer and a "Yes" answer. A single counter example is sufficient in the No-case. The "Yes" must be justified in terms of properties or previously accepted statements.

1. (a) a is less than b.
   (b) x is greater than y.
   (c) m is either greater than or equal to 3; or, m is not less than 3.
   (d) n is either less than or equal to 3; or, n is not greater than 3.
   (e) 0 is less than 1, and 1 is less than 2; or, 1 lies between 0 and 2.
   (f) x is greater than zero; or, x is positive.

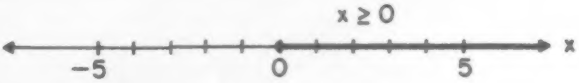
(g) 5 is greater than or equal to x, and x is greater than or equal to -5; or, x is between -5 and 5, including -5 and 5; or x is not less than -5 nor greater than 5.

(h) Same as (g).

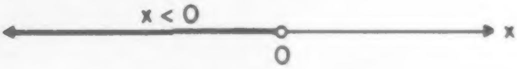
2. (a) $k > 0$
   (b) $r < 0$
   (c) $t \leq 0$
   (d) $s \geq 0$
   (e) $2 < g < 3$
   (f) $2 \leq w \leq 3$
   (g) $a < w < b$

3. (a) $x > -2$ and $5 > x$; 6
   (b) $n \geq 3$ and $11 > n$; 4
   (c) $y \geq 0$ and $7 \geq y$; 8

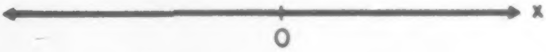
4. (a)



(b)

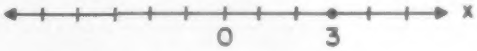


(c)



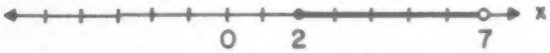
Since for all x, either $x = 0$, $x > 0$ or $x < 0$, the union of $x \geq 0$ and $x < 0$ includes all real numbers.

(d)



$x = 3$ is the intersection of $x \geq 3$ and $x \leq 3$.

(e)



5. (a) $15z > 0$ by the hypothesis.

$\frac{1}{15} \cdot 15z > \frac{1}{15} \cdot 0$ by the multiplicative property of order using a positive multiplier.

$z > 0$ by simplifying names.

Thus the required set includes all numbers greater than zero or all positive numbers.

(b) No. If $x = 8$, $y = 3$ and $z = 5$, the conditions are satisfied and $z > y$.

If $x = 10$, $y = 7$, $z = -2$, the conditions are satisfied and $y > z$.

Thus z may be $> y$ or y may be $> z$.

6. (a) If $3m > 2$ then $-2 > -3m$ by multiplicative property of order with (-1) as multiplier.

But $-2 > -3m$ means the same as $-3m < -2$.

(b) The additive property of order. Add $(-x)$.

(c) The multiplicative property of order. Multiply by $\frac{1}{5}$.

(d) The additive property of order. Add $(-x - 7)$.

(e) The multiplicative property of order with (-1) as multiplier.

(f) The multiplicative property of order using p (a positive number) as the multiplier.

(g) The multiplicative property of order using m (a negative number) as the multiplier.

7. (a) It does follow.

(b) It does follow.

(c) It does not follow since the first statement gives a possibility of $9x = 15$ in which case $x = \frac{5}{3}$ and this is not included in the second statement.

(d) It does not follow. $2x = 11$ (part of the second statement) could only follow the first if $5x - 4 = 3x + 7$ were included in the first statement which is not the case.

(e) and (f) both follow. The properties of order and/or equality may be applied to each of the two parts of the statement separately.

Problem Set 3-4

Problems 3, 4 and 5 give a foundation for the work on the coordinate system presented in the next section.

1. (a) $A'Q = 1$; $PQ = \frac{1}{2}$; $PR = \frac{3}{2}$
$PS = \frac{5}{2}$; $PT = 2\frac{3}{4}$; $ST = \frac{1}{4}$

(b) The distances are the same as in (a) since the distance from A to A' is the same as the distance from Q to R.

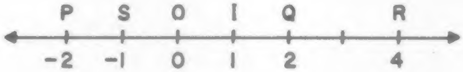
(c) QR (relative to $\{A, A'\}$) = 1
Distances in (a) and (b) are the same.
This illustrates Postulate 11.

(d) (1) $\frac{3}{2}$ (2) $\frac{3}{2}$ (3) $\frac{3}{8}$
(4) 6 (5) 3 (6) 1

2. (a)
$PC \leftrightarrow 2$
$PD \leftrightarrow 2\frac{1}{2}$
$PM \leftrightarrow 4$
$C \leftrightarrow 2$
$D \leftrightarrow 2\frac{1}{2}$
$M \leftrightarrow 4$

(b)
$PM \leftrightarrow \frac{8}{3}$
$QM \leftrightarrow 2$
$CM \leftrightarrow \frac{4}{3}$
$C \leftrightarrow \frac{4}{3}$
$Q \leftrightarrow 2$
$P \leftrightarrow \frac{8}{3}$

*3. (a)



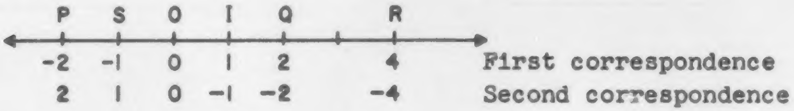
(b) Relative to $\{Q, I\}$
PQ = 4
SR = 5
PR = 6
RO = 4
QR = 2

(c) IQ (relative $\{O, I\}$) = 1

(d) By Postulate 11 the distances (relative to {I,Q}) are:
PQ = 4
SR = 5
PR = 6
RO = 4
QR = 2

(e) PQ = 1 - (-3) = 4
SR = 3 - (-2) = 5
PR = 3 - (-3) = 6
RO = 3 - (-1) = 4
QR = 3 - 1 = 2
These distances are the same as the corresponding distances in (d).

*4. (a)


P
S
O
I
Q
R
-2
-1
0
1
2
4
First correspondence
2
1
0
-1
-2
-4
Second correspondence

(b) PR (relative to {O,I}) = 6
PR (relative to {O,S}) = 6
The distances are equal. We would expect this since OS (relative to {O,I}) = 1.

*5. (a) Yes. The point corresponding to 1 is $\frac{1}{8}$ of the distance AB to the right of A. This is a unique point.

(b) Yes The point corresponding to 1 is $\frac{1}{5}$ of the distance AC to the left of A. It is a unique point.

Problem Set 3-5

Problems 9, 10 and 11 introduce the absolute value symbol. Many teachers and students appreciate its convenience and may find it useful in various considerations in this course. It will be discussed in the text in Chapter 8.

1. (a) P is origin, Q is unit-point
   (b) $PQ = 1$ $XY = y - x$
   $BC = 4.3$ $CB = 4.3$
   $AX = x + 4$ $YX = y - x$
   $CY = y - 1.8$

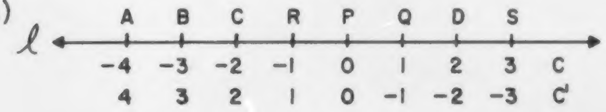
2. The student can only select as his second coordinate system the system with coordinates $x' = -x$. Therefore, if the coordinate assigned to Q by the first system is q, then the coordinate assigned by the second would be -q. The distance PQ is the same in both systems.

3. (a) Yes by Theorem 3-1. It is unique.
   (b) Yes, by the Ruler Postulate. It is unique for any selected unit-pair.

4. (a) No.
   (b) No.
   (c) In part (a) P separates the line into two sets. In part (b) Q separates the line into two sets.

5. (a) $17 \frac{1}{2}$
   (b) $x - y$
   (c) $y - x$
   (d) 0
   (e) $a - b$ if $a > b$
   $b - a$ if $b > a$
   0 if $a = b$

6. (a)



(b) (1) P is the origin in both systems.

(2) {P,Q} is the unit-pair in C, {P,R} is the unit-pair in C'.

PQ (relative to {P,R}) = 1 = PR (relative to {P,Q}).

(3) The unit-point is to the right of the origin in C and to the left of the origin in C'.

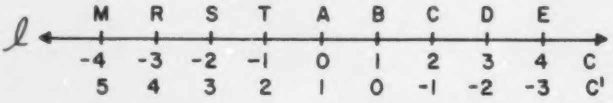
(c) $x' + x = 0$ or $x' = -x$

(d) BR in $C = -1 - (-3) = 2$

BR in $C' = 3 - 1 = 2$

They are equal.

7. (a)



(b) A is the origin in C and B the unit-point.

B is the origin in C' and A the unit-point.

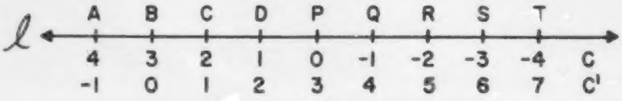
The origin and unit-point have been interchanged in the two systems so the unit-point is to the right of the origin in C and to the left of the origin in C'.

(c) $x' + x = 1$ or $x' = -x + 1$

(d) BR in $C = 1 - (-3) = 4$; BR in $C' = 4 - 0 = 4$.

They are equal.

8. (a)



(b) P is the origin in system C. B is the origin in system C'. B is to the left of P. {P,D} is the unit-pair in C. {B,C} is the unit-pair in C'.

PD (relative to {B,C}) = 1 = BC (relative to {P,D}).

The unit-point is to the left of the origin in C and to the right of the origin in C'.

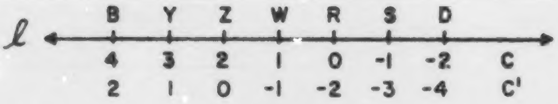
(c) $x' + x = 3$, or $x' = -x + 3$.

(d) BR in system $C = 3 - (-2) = 5$

BR in System $C' = 5 - 0 = 5$.

They are equal.

9. (a)



(b) (1) R is the origin in C, while Z is the origin in C' and Z is to the left of R.

(2) [R,W] is the unit-pair in C.
[Z,Y] is the unit-pair in C'. RW (relative to [Z,Y]) = 1 = ZY (relative to [R,W])

(3) In both systems, the unit-point is to the left of the origin.

(c) $x - x' = 2$ or $x' = x - 2$

(d) BR in $C = 4 - 0 = 4$
BR in $C' = 2 - (-2) = 4$
They are equal.

10. (a) (1) 5
(2) 5
(3) 1
(4) 1
(5) $|-1 - 1| = |-2| = 2$

(b) In each of these problems if the order in the expressed difference were changed, the sign of the expression within the absolute value symbol would be changed. The final expressions using the two orders would, of course, be equivalent.
(1) $|x - y|$
(2) $|(a + b) - b| = |a|$
(3) $|(a + b) - (a - b)| = |2b| = 2|b|$
(4) $|(a + b) - (b + a)| = |0| = 0$
(5) $|(a - b) - (b - a)| = |2a - 2b| = 2|a - b|$.

Problem Set 3-6

Problems 5 and 6 begin to build toward an understanding of the relation between coordinates in two coordinate systems established on a given line. They suggest the conclusion that when the sum of corresponding coordinates is a constant, the scales in the two systems appear to "run in opposite directions" and when the difference between corresponding coordinates is a constant, the scales appear to "run in the same direction."

1.

	Origin	Unit-Point
(a)	C	F
(b)	B or G	G or B
(c)	E	D
(d)	C or F	F or C
(e)	B	E
(f)	C	A
(g)	B or E	E or B
(h)	D or E	E or D

There is a choice in the case of a segment.

2. (a) $\overline{CF}$
(b) $\overrightarrow{CF}$
(c) C
(d) ℓ
(e) $\overline{BE}$
(f) $\overline{BF}$
(g) $\overline{CE}$
(h) Empty set

3. (a) $\overrightarrow{MR}$ (b) $\overrightarrow{MS}$ (c) $\overline{MR}$ or $\overline{RM}$

4. (a) Yes
(b) Between 0 and 1
(c) $x - 0 = 1 - x$
(d) $\{\frac{1}{2}\}$; only one
(e) One, since a coordinate system is a one-to-one correspondence.

*5. $x' = -x + 2$; yes

*6. $x' = x - 3$; the rulers are placed against ℓ so that their scales "run in the same direction," whereas in Problem 5 the scales "run in opposite directions."

Problem Set 3-7

These problems are primarily to prepare for the next section. They emphasize that (a) the relation between the measures of a given distance relative to two different units is determined by the relation between those two units and (b) the ratio between the measures of two different distances is independent of the unit used. Help students to see these relationships in the table in Problem 2.

1. (a) 0
   (b) $x > 0$
   (c) $x \leq 0$
   (d) C is 5
       D is 4
       E is $\frac{1}{2}$
   (e) $0 < p < 1$
   (f) (1) No
       (2) Yes
       (3) Yes
       (4) No
       (5) No
       (6) No
   (g) interior, ray, opposite

*2.

	PQ	RS	TV	VW
inches	3	6	18	24
feet	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{2}$	2
yards	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{2}{3}$
alphas	2	4	12	16
betas	8	16	48	64

*3. (a) 3
(b) 3
(c) 36
(d) $\frac{1}{36}$
(e) 12
(f) $\frac{1}{12}$
(g) $\frac{1}{3}$

4. (a) smaller
(b) 4
(c) The first pair of points is 4 times as far apart as the second pair of points.

*5. (a) The answer is given.
(b) 8 , 8
(c) 8 , 8
(d) 10

*6. (a) 12
(b) $\frac{1}{12}$
(c) 3
(d) $\frac{1}{3}$
(e) 36
(f) $\frac{1}{36}$

*7. (a) 12
(b) $\frac{1}{12}$
(c) 36

Problem Set 3-8

1. (a) 3 , $\frac{1}{2}$, 1 , 2
(b) G , B , E , G
(c) $\frac{\frac{1}{2}}{2} = \frac{\frac{1}{4}}{1} = \frac{1}{4}$
(d) $\frac{2}{1} = \frac{\frac{2}{3}}{\frac{1}{3}}$

2. (a) (1) $\frac{-1-(-2)}{1-0} = 1$

(2) 1

(3) $\frac{-1-e}{1-(-3)} = \frac{-1-e}{4}$

(4) $\frac{-1-e}{4} = 1$

$e = -5$

(5) $\frac{x'-(-2)}{x-0} = 1$

$x' = x - 2$

(6) $x' = -5 - 2$

$x' = -7$

(b) (1) $\frac{15-9}{5-3} = 3$

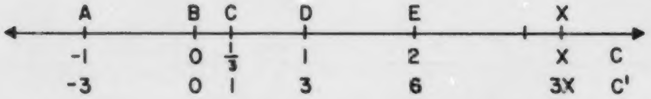
(2) 3

(3) $\frac{d-9}{11-3} = 3$; $d = 33$

(4) $\frac{9-y}{3-x} = 3$; $y = 3x$

(5) $y = 3(-8) = -24$

*3. ℓ



(a) Yes. Since C is a one-to-one correspondence, for each point X in ℓ there is a unique number x in C. For each number x in C there is a unique number 3x in C'. Thus for each point X in ℓ there is a unique number 3x in C'.

(b) If we select special points such as A and D, using the numbers -3 and 3 assigned to them respectively by C', we find that, since AD (relative to {B,C}) = 6 = 3 - (-3), C' appears to satisfy the second criterion for a coordinate system. A more general consideration follows.

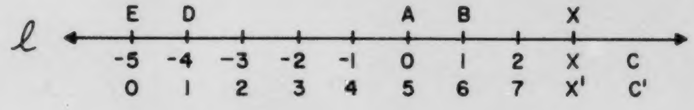
To show that for every pair of points on ℓ correspondence C' satisfies the second criterion for a coordinate system, assume X_1 and X_2 be any two points on ℓ with coordinates in C x_1 and x_2 respectively and with $x_2 > x_1$. The numbers associated with X_1 and X_2 by correspondence C' are, then, $3x_1$ and $3x_2$ respectively and by the multiplicative property of order $3x_2 > 3x_1$. If C' is a coordinate system, its unit-pair is {B,C} and we must show that X_1X_2 (relative to {B,C}) $= 3x_2 - 3x_1$.

X_1X_2 (relative to $\{B,C\}$) = $3X_1X_2$ (relative to $\{B,D\}$) since BD (relative to $\{B,C\}$) = 3. But, since C is a coordinate system, X_1X_2 (relative to $\{B,D\}$) = $x_2 - x_1$. Hence X_1X_2 (relative to $\{B,C\}$) = $3(x_2 - x_1)$ = $3x_2 - 3x_1$. It follows, since C' satisfies both conditions in the definition, that correspondence C' is a coordinate system.

(c) Yes

(d) No

*4.



$$x' = x + 5$$

(a) C' is a one-to-one correspondence.

(b) If C' is a coordinate system, $\{E,D\}$ is the unit-pair in C'. Let X_1 and X_2 be points on ℓ with coordinates in system C, x_1 and x_2 respectively and such that $x_2 > x_1$. Then the numbers assigned to X_1 and X_2 by C' are $x_1 + 5$ and $x_2 + 5$ respectively. By the addition property of order $x_2 + 5 > x_1 + 5$.
X_1X_2 (relative to $\{E,D\}$) = X_1X_2 (relative to $\{A,B\}$) since ED (relative to $\{A,B\}$) = 1. But since C is a coordinate system, X_1X_2 (relative to $\{A,B\}$) = $x_2 - x_1$ which, by changing the form, $= (x_2 + 5) - (x_1 + 5)$. Thus X_1X_2 (relative to $\{E,D\}$) $= (x_2 + 5) - (x_1 + 5)$.
Again, as in Problem 3(b), the compliance with the definition of a coordinate system may easily be noted for any two specific points. It follows that the correspondence C' is a coordinate system since it satisfies both conditions stated in the definition.

(c) and (d) Neither the origin nor the unit-point are the same for the system C and C'.

5. Yes.

$$\frac{PQ\ (\text{relative to } \{B,B'\})}{PQ\ (\text{relative to } \{A,A'\})} =$$

$$\frac{RS\ (\text{relative to } \{B,B'\})}{RS\ (\text{relative to } \{A,A'\})} =$$

$$\frac{TV\ (\text{relative to } \{B,B'\})}{TV\ (\text{relative to } \{A,A'\})} = k$$

If $PQ + RS = TV$ (relative to $\{A,A'\}$) , then
$k \cdot PQ$ (relative to $\{A,A'\}$) + $k \cdot RS$ (relative to $\{A,A'\}$) =
$k \cdot TV$ (relative to $\{A,A'\}$). This is the same as
PQ (relative to $\{B,B'\}$) + RS (relative to $\{B,B'\}$) =
TV (relative to $\{B,B'\}$).
Thus $PQ + RS = TV$ (relative to $\{B,B'\}$).

Problem Set 3-9

1. (a) RS (in C') = 4 ; RS (in C) = 2 ; RS (in C') = 2RS (in C)
   (b) RT (in C) = 1
   (c) RT (in C') = 2
   (d) 2 or -2
   (e) ST (in C) = 3 ; ST (in C') = 6
   (f) -2 or 10
   (g) Yes. The coordinate of T in C' must be -2 since it must satisfy both distance relations.

2. (a) $a = 2$, $b = 0$. Therefore $x' = 2x$.
   (b) If $x = -1$, $x' = 2(-1) = -2$.

3. (a) RS (in C') = 3 , RS (in C) = 6 , RS (in C') = $\frac{1}{2}$ RS (in C)
   (b) RT (in C) = 1
   (c) RT (in C') = $\frac{1}{2}$
   (d) $\frac{3}{2}$ or $\frac{1}{2}$
   (e) ST (in C) = 5 , ST (in C') = $\frac{5}{2}$
   (f) $\frac{1}{2}$ or $-4\frac{1}{2}$
   (g) The coordinate of T in C' must be $\frac{1}{2}$ since $\frac{1}{2}$ satisfies both distance relations.

4. $1 = a \cdot 2 + b$ Then $1 = (-\frac{1}{2})2 + b$

$\underline{-2 = a \cdot 8 + b}$ $b = 2$

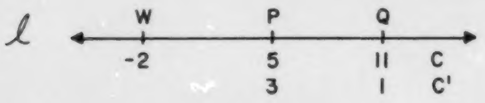
$-3 = a \cdot 6$

$a = -\frac{1}{2}$ Thus $x' = -\frac{1}{2}x + 2$

If $x = 3$, $x' = \frac{1}{2}(3) + 2 = \frac{1}{2}$

The coordinate of T (in C') is $\frac{1}{2}$.

5.



(a) (1) PQ (in C) = 6 , PQ (in C') = 2 ,

PQ (in C') = $\frac{1}{3}$ PQ (in C)

(2) PW (in C) = 7

(3) PW (in C') = $\frac{7}{3}$

(4) The coordinate of W (in C') is either $3 + \frac{7}{3}$ or $3 - \frac{7}{3}$, that is, $5\frac{1}{3}$ or $\frac{2}{3}$.

(5) QW (in C) = 13 , QW (in C') = $\frac{13}{3}$.

(6) The coordinate of W (in C') is either $1 + \frac{13}{3}$ or $1 - \frac{13}{3}$; either $5\frac{1}{3}$ or $-4\frac{1}{3}$.

(7) The coordinate of W (in C') must be $5\frac{1}{3}$.

(b) $3 = 5a + b$ Thus $x' = -\frac{1}{3}x + \frac{14}{3}$

$\underline{1 = 11a + b}$

$2 = -6a$ If $x = -2$, $x' = -\frac{1}{3}(-2) + \frac{14}{3}$

$a = -\frac{1}{3}$

$3 = 5(-\frac{1}{3}) + b$ $x' = 5\frac{1}{3}$

$b = 4\frac{2}{3}$ The coordinate of W (in C') = $5\frac{1}{3}$.

6. Parts (a) and (b) can be solved in the manner of 5(a) above. However, students may prefer the method of 5(b), in which case they will solve part (c) before (a) and (b).

(c) $32 = a \cdot 0 + b$ (a) $F = 1.8(40^\circ) + 32^\circ$

$\underline{212 = a \cdot 100 + b}$ $F = 104^\circ$

$b = 32$ (b) $0 = 1.8C + 32^\circ$

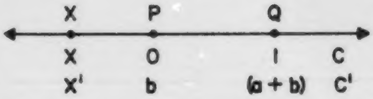
$a = 1.8$ $C = -17\frac{7}{9}^\circ$

$F = 1.8C + 32$

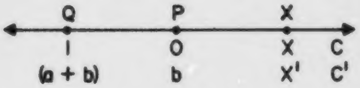
7. (a) $x' = b$

(b) $x' = a + b$

(c)



or



(d) PQ (in C) $= 1 - 0 = 1$

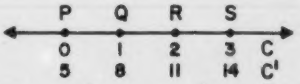
PQ (in C') $= (a + b) - b = a$, if $(a + b) > b$

PQ (in C') $= b - (a + b) = -a$, if $b > (a + b)$

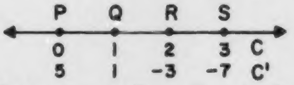
(e) If $(a + b) > b$ which means if $a > 0$, then the coordinates C' will *increase* from b to $(a + b)$ as we move from P to Q. The coordinates in scale C *increase* from 0 to 1 as we move from P to Q. Thus the scales "run in the same direction."

If $(a + b) < b$ or $a < 0$, then the coordinates in C' will *decrease* from b to $(a + b)$ as we move from P to Q whereas the coordinates in scale C *increase* from 0 to 1 as we move from P to Q. Thus the scales "run in the opposite direction."

If $b = 5$ and $a = 3$, then



If $b = 5$ and $a = -4$, then

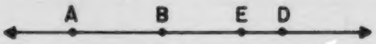


(f) If $a = 0$, then $a + b = b$ and in the diagram in (C) above two points P and Q would both have the coordinate b. This would not give a one-to-one correspondence and C' would not be a coordinate system.

Explained in another way: if in $x' = ax + b$, $a = 0$ then $x' = b$ for all values of x. Thus a single number b is associated with every point on line ℓ. This is not a coordinate system.

*8. (a) $AE = 5 + \frac{2}{3}(9 - 5) = 7\frac{2}{3}$ miles

(b)



$AE = x$, $AB = x_1$, $AD = x_2$; $BE = \frac{2}{3} BD = \frac{2}{3}(AD - AB)$

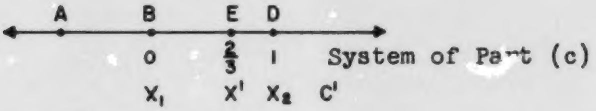
$AE = AB + \frac{2}{3} BD$

$x = x_1 + \frac{2}{3}(x_2 - x_1)$

(c) If x is the coordinate of E , then

$$x - 0 = \frac{2}{3}(1 - 0) \text{ or } x = \frac{2}{3} .$$

(d)



From $x' = ax + b$

$$x_1 = a \cdot 0 + b$$

$$x_2 = a \cdot 1 + b$$

$$b = x_1 ; a = x_2 - x_1$$

$$x' = (x_2 - x_1)x + x_1$$

$$x' = (x_2 - x_1)\frac{2}{3} + x_1$$

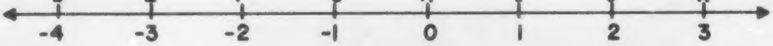
or $x' = x_1 + \frac{2}{3}(x_2 - x_1)$.

Problem Set 3-10

1. (a)
(1) $0 \leq x \leq 8$
(2) $2 \leq x \leq 6$
(3) $-4 \leq x \leq 2$
(4) $x \geq 0$
(5) $x \leq 8$
(6) $x \leq 6$
(7) $x \geq 2$
(8) $x \leq 2$
(9) $x \geq -4$
(10) $x \leq 0$
(11) $x \geq 0$
(12) $x \geq 2$

(b)
(4) and (11) , $\overrightarrow{GK}$ and $\overrightarrow{GI}$
(7) and (12) , $\overrightarrow{HJ}$ and $\overrightarrow{HK}$

2.


D
E
F
G
H
I
J
K
-4
-3
-2
-1
0
1
2
3

(a) (1) $-1 \leq x \leq 3$

(2) $0 \leq x \leq 2$

(3) $-3 \leq x \leq 0$

(4) $x \geq -1$

(5) $x \leq 3$

(6) $x \leq 2$

(7) $x \geq 0$

(8) $x \leq 0$

(9) $x \geq -3$

(10) $x \leq -1$

(11) $x \geq -1$

(12) $x \geq 0$

(b) Same as 1(b)

3. $x = x_1 + k(x_2 - x_1)$

(a) $x = 2 + 4k$

(b) $x = -2 + 6k$

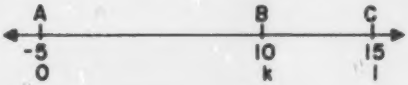
(c) $x = 3 + k(-7) = 3 - 7k$

(d) $x = -15 + 30k$

(e) $x = -5 + 5k$

4. Two methods of solution are shown for each problem.

(a)


A
B
C
-5
10
15
0
k
1

$x = x_1 + k(x_2 - x_1)$

$10 = -5 + k(15 - [-5])$

$k = \frac{3}{4}$

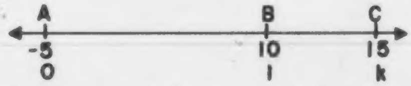
Yes, Yes, No

$$\frac{10 - (-5)}{k - 0} = \frac{15 - (-5)}{1 - 0}$$

$$\frac{15}{k} = 20$$

$$k = \frac{3}{4}$$

(b)


A
B
C
-5
10
15
0
1
k

$x = -5 + k(10 - [-5])$

$15 = -5 + 15\ k$

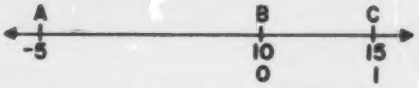
$k = \frac{4}{3}$

No, Yes, No

$$\frac{15 - (-5)}{k - 0} = \frac{10 - (-5)}{1 - 0}$$

$$\frac{20}{k} = \frac{15}{1}$$

$$k = \frac{4}{3}$$

(c) 

$x = 10 + k(15 - 10)$

$-5 = 10 + 5k$

$k = -3$

No, No, Yes

$\frac{10 - (-5)}{0 - k} = \frac{15 - 10}{1 - 0}$

$\frac{15}{-k} = 5$

$k = -3$

5. (a) $5 + \frac{1}{2}(11 - 5) = 8$ or $\frac{5 + 11}{2} = 8$

(b) $-2 + \frac{1}{2}[-9 - (-2)] = -5\frac{1}{2}$ or $\frac{-9 + (-2)}{2} = -5\frac{1}{2}$

(c) $\frac{1}{2} + \frac{1}{2}(\frac{2}{3} - \frac{1}{2}) = \frac{7}{12}$ or $\frac{\frac{2}{3} + \frac{1}{2}}{2} = \frac{7}{12}$

(d) $x = x_2 + \frac{1}{2}(x_1 - x_2) = \frac{x_1 + x_2}{2}$

$x = x_1 + \frac{1}{2}(x_2 - x_1) = \frac{x_1 + x_2}{2}$

$x = \frac{x_2 + x_1}{2}$

(e) $x = -r + \frac{1}{2}[(r + s) - (-r)] = \frac{s}{2}$ or

$x = (r + s) + \frac{1}{2}[-r - (r + s)] = \frac{s}{2}$ or

$x = \frac{(r + s) - r}{2} = \frac{s}{2}$

6. $7 = 4 + \frac{1}{2}(x_2 - 4)$ or $7 = \frac{4 + x_2}{2}$

$x_2 = 10$ $x_2 = 10$

7. $-7 = x + \frac{1}{2}(-2 - x)$ or $\frac{-2 + x}{2} = -7$

$x = -12$ $x = -12$

or $-2 - (-7) = -7 - x$

$x = -12$

8. (a) $x_1 = 3 + \frac{1}{3}(12 - 3)$; $x_2 = 3 + \frac{2}{3}(12 - 3)$

$x_1 = 6$ $x_2 = 9$

(b) $x_1 = -1 + \frac{1}{3}[x - (-1)]$; $x_2 = -1 + \frac{2}{3}[4 - (-1)]$

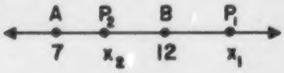
$x_1 = \frac{2}{3}$ $x_2 = \frac{7}{3}$

(c) Trisection point nearer to x_1 has coordinate = $x_1 + \frac{1}{3}(x_2 - x_1)$ and
Trisection point nearer to x_2 has coordinate = $x_1 + \frac{2}{3}(x_2 - x_1)$.

9. (a) $x = 7 + \frac{1}{4}(5) = 8\frac{1}{4}$

(b) $x = 7 + \frac{3}{4}(5) = 10\frac{3}{4}$

(c)



Either P has coordinate x_1 such that $x_1 - 12 = \frac{1}{4}(12 - 7)$ and $x_1 = 13\frac{1}{4}$ or P has coordinate x_2 such that $12 - x_2 = \frac{1}{4} \cdot 5$ and $x_2 = 10\frac{3}{4}$.

10.

	k	x	P
(a)	$k = 0$	$x = -5$	A
(b)	$k = 1$	$x = 7$	B
(c)	$k = \frac{1}{2}$	$x = 1$	midpoint of $\overline{AB}$
(d)	$k = \frac{1}{3}, \frac{2}{3}$	$x = -1, 3$	trisection points of $\overline{AB}$
(e)	$k = 2$	$x = 19$	P on $\overrightarrow{AB}$ such that $AP = 2AB$
(f)	$0 \leq k \leq 1$	$5 \leq x \leq 7$	$\overline{AB}$
(g)	$k \geq 0$	$x \geq -5$	$\overrightarrow{AB}$
(h)	$0 < k < 1$	$-5 < x < 7$	interior of $\overline{AB}$
(i)	$k < 0$	$x < -5$	interior of ray opposite to $\overrightarrow{AB}$
(j)	$k = 5$	$x = 55$	P is on $\overrightarrow{AB}$ and $AP = 5 \cdot AB$
(k)	$k = -5$	$x = -65$	P is on ray opposite to $\overrightarrow{AB}$ and $AP = 5 \cdot AB$

Problem Set 3-11

1. (a) XB = 3 , AF = 3 , GD = 2 , FG = 7

(b) DB = 7 , Distance from D to B = 7 , DB = 7 , BD = 7
All answers are the same since all are symbols for the same measure.

2. (a) True. $1 - 0 = 1$.
   (b) False. $3 - 1 = 2$ not 3.
   (c) True. $3 - 1 = 2$.
   (d) False since they are different sets of points.
   (e) True. $7 - (-1) = 8$ and $6 - (-2) = 8$.
   (f) False since two segments are congruent only if their lengths have the same measure.
   $7 - (-1) \neq 0 + (-3)$.

3. (a) True
   (b) Not meaningful
   (c) Not meaningful
   (d) True
   (e) False
   (f) True
   (g) Not meaningful
   (h) False
   (i) Not meaningful
   (j) True

4. $MT = MS + ST$; Theorem 3-9

5. No. If P, Q, R are collinear in that order, then Q is between P and R. This means that the coordinate of Q is between the coordinates of P and R. Since $-5 < -2 < 3$, the coordinate of Q must be -2.

6. $AH = 5$.

7. (a) With 9 , yes. With -5 , no. Every point on $\overrightarrow{DE}$ has a non-negative coordinate
   (b) (1) 7
       (2) $\frac{1}{3}$
       (3) 10
       (4) 8
       (5) 11
       (6) 2 or 0

(7) 7

(8) k must be ≥ 0 since EF and EG are both non-negative.

f must be ≥ 0 since F is in $\overrightarrow{DE}$.

If $f > 1$, which means F is to the right of E, then

$$EF = f - 1 = k \cdot 2 \text{ or } f = 2k + 1.$$

Then $2k + 1 > 1$ and $k > 0$.

Thus for all $k > 0$, $f = 2k + 1$ and F is in $\overrightarrow{DE}$ such that E is between D and F.

If $0 \leq f \leq 1$, then $EF = 1 - f = 2k$ or $f = 1 - 2k$.

Then $0 \leq 1 - 2k \leq 1$

$-1 \leq -2k \leq 0$

$\frac{1}{2} \geq k \geq 0$

Thus for all k such that $0 \leq k \leq \frac{1}{2}$, $f = 1 - 2k$ and F is in $\overline{DE}$.

If $0 < k \leq \frac{1}{2}$, f has 2 values and F has 2 positions in $\overrightarrow{DE}$.

If $k > \frac{1}{2}$ or if $k = 0$, f has only one value.

8. (a) (1) 12
   (2) 5
   (3) $\frac{3}{2}$

   (b) (1) $\frac{5}{7}$
   (2) 3

   (c) -4 or 0

9. (a) (1) x
   (2) $18 - x$
   (3) x
   (4) $x - 18$
   (5) $18 - x$

   (b) (1) $0 \leq x \leq 18$
   (2) $x \geq 18$
   (3) $x \leq 0$

10. Let p, q, r be coordinates of P, Q, R respectively. The order of collinearity given demands that either $p < q < r$ or $r < q < p$

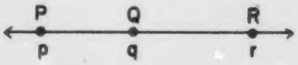
(a) In case $p < q < r$

$q - p < r - p$

or $PQ < PR$

also $r - q < r - p$

or $QR < PR$



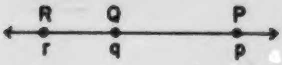
(b) In case $r < q < p$

$q - r < p - r$

or $RQ < PR$

also $p - q < p - r$

or $PQ < PR$



Chapter 3

Review Problems

1. (a) I or V (c) R or V or J (e) N or J or empty set
   (b) N (d) The empty set

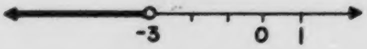
2. (a) positive
   (b) a positive number less than 2
   (c) negative

3. (a) T (d) N
   (b) T (e) N
   (c) F

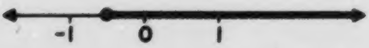
4. (a) N (c) T
   (b) T (d) T

5. (a) $x < -3$ (c) $x > -14$ (e) $x \geq -.5$
   (b) $x < \frac{7}{3}$ (d) $x < -\frac{3}{2}$

6. (a)



(b)



7. (a) 8 (e) 11.1
   (b) 8 (f) $-7b$
   (c) 7 (g) $2b$
   (d) 9 (h) If $y_2 \geq y_1$, then $y_2 - y_1$
   If $y_2 < y_1$, then $y_1 - y_2$
   (i) $-2b$

8: b, e

9. ray (d, j), point (b, e), line (h), segment (f)

10. (a) $\frac{3}{2}$ (b) $\frac{13}{2}$ (c) $\frac{r + s}{2}$

11. (a) 11 (b) -1 (c) 2

12. (a) $\frac{6}{3}$ or 2 ; $\frac{5}{2\frac{1}{2}}$ or 2

(b) $\frac{6}{5}$; $\frac{3}{2\frac{1}{2}} = \frac{6}{5}$

(c) 2

(d) They are equal. Postulate 13.

(e) They are equal. Theorem 3-4.

13. (a) False

(b) Not meaningful

(c) True

(d) False

(e) True

14. (a) Yes $\overrightarrow{RW}$ is the set of points with coordinates $x \geq 0$ and 17 satisfies this requirement.

(b) Yes, from the definition of a coordinate system on a line.

(c) (1) 12 , (2) 8 , (3) $p - 1 = 5$ or $p = 6$

(4) If $WP = k \cdot RW$, then $WP = k(1 - 0) = k$. k must be ≥ 0 since WP is a distance and must be non-negative.

If $k = 0$, then $P = W$ and the coordinate of $P = 1$.

If $k > 1$, then W is between R and P and $p = 1 + k$.

If $0 \leq k \leq 1$, there are two possible positions of P , one with W between R and P in which case $p = 1 + k$, the other when P is between R and W and $p = 1 - k$.

(d) In parts 1, 2, and 3 of (c) only one answer is possible. In part 4 of (c) more than one answer is possible as indicated above.

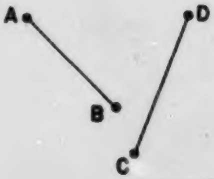
(e) Let p be the coordinate of P.

Then $RP = p - 0$ or $0 - p$
and $WP = p - 1$ or $1 - p$.
The solutions from (c) would then be

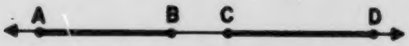
(1) $p = 12$ or $p = -12$
(2) $p = 8$ or $p = -8$
(3) $p = 6$ or $p = -4$
(4) $p = 1 + k$ or $p = 1 - k$ with $k \geq 0$

15. (a) T (d) T (g) T (j) T
(b) T (e) F (h) T
(c) T (f) T (i) T

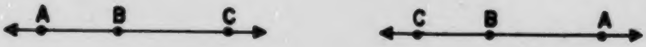
16. (a)



(b)



(c)



(1) B ; $\overline{BC}$
(2) $\overline{AC}$; $\overline{AC}$
(3) The empty set
(4) The interior of $\overline{AC}$

17. (a) $AB + BC = AC$
(b) If $AB = BC$

18. (a) Theorem 3-7 , The Betweenness-Coordinates Theorem.

(b) Theorem 3-7 , The Betweenness-Coordinates Theorem.

(c) Theorem 3-9 , The Betweenness-Distance Theorem since S is between R and T .

19. PR (in cm) = 30
QR (in cm) = 100 QR (in m) = 40
QR (in cm) = (.1) QR (in mm) = 10

Let C be a (cm) coordinate system on the given line which assigns 0 to Q and 40 to R . Then, if p is the coordinate of P , either $p - 40 = 30$ or $40 - p = 30$ and p is either 10 or 70 . If $p = 70$, then $PQ = 70$. But this contradicts the hypothesis that $QP = 10$, therefore $p = 10$. Since $0 < 10 < 40$, P is between Q and R by the Betweenness-Coordinate Theorem.

20. (a) 3 , 10 (c) 10
(b) 3 (d) 10

21. $\overrightarrow{BA}$ since $x > 4$

22. (a) $x \geq -3$ (b) $y \geq -3$ (c) $z \leq 31$

23. (a) $p = \frac{a + b}{2}$

(b) $p = a + \frac{1}{3}(b - p)$ or $p = a + \frac{1}{4}(b - a) = \frac{3a + b}{4}$

(c) $p = a + \frac{2}{5}(b - a)$

(d) $p = a + k(b - a)$

(e) No. $k = \frac{PA}{AB}$ and since PA and AB , being distances, are both positive, their quotient is positive. k cannot be ≥ 1 since this would mean $PA \geq AB$. But P is between A and B so $AP + PB = AB$ and, since $PB > 0$, $PA < AB$. $k \neq 0$ since it would then follow that $PA = 0$ and $P = A$ which contradicts the hypothesis that P and A are distinct points.

24. (a) Relative to the same unit pair the distance between P and Q is the same as the distance between M and N .

(b) The segments are congruent, i.e. they are equal in length.

(c) The symbols are names for the same line. The set of points are equal.

(d) The distance QN is greater than the distance QM.

(e) $\overrightarrow{QN}$ and $\overrightarrow{QM}$ are names for the same ray with endpoint Q and containing points N and M .

25. (a) Yes.

(b) The union is $\overleftrightarrow{BC}$; the intersection is A .

(c) The union is $\overleftrightarrow{AB}$; the union is $\overrightarrow{AB}$.

(d) The intersection is $\overline{AB}$; the union is $\overrightarrow{AB}$.

26. (a) $x' = 1 - \frac{1}{2}x = \frac{2 - x}{2}$.

27.

Coordinate System	Relationship	Coordinate of D	Coordinate of E	Coordinate of F	Coordinate of G	Coordinate of H	Coordinate of I
First	x	0	3	2	-1	5	-2
Second	$x' = 6x$	0	18	12	-6	30	-12
Third	$x'' = x - 3$	-3	0	-1	-4	2	-5
Fourth	$x''' = -2x + 2$	2	-4	-2	4	-8	6

28. (a) $x_1 \leq k \leq x_2$

(b) $k \geq x_1$

(c) $k \leq x_1$

(d) $k \leq x_2$

(e) k is real.

29. We have not proved that $AB + BC = AC$ implies that B is between A and C. This is the converse of the Betweenness-Distance Theorem but the truth of the converse of a theorem does not necessarily follow from the truth of the theorem. The text stated that this converse could be proved.

30. (a)

$$x' = ax + b$$
$$-3 = a \cdot 3 + b$$
$$13 = a(-5) + b$$
$$a = -2$$
$$b = 3$$
$$x' = -2x + 3$$

(b)

$$s = -2 \cdot 0 + 3$$
$$s = 3$$

31. $x' = x_1 + k(x_2 - x_1)$

(a) $x' = x_1 + 0\ (x_2 - x_1) = x_1$

(b) $x' = x_1 + (x_2 - x_1) = x_2$

(c) $x' = x_1 + \frac{1}{2}\ (x_2 - x_1) = \dfrac{x_1 + x_2}{2}$

(d) $x' = x_1 - 2\ (x_2 - x_1) = 3x_1 - 2x_2$

Chapter 4

ANSWERS AND SOLUTIONS

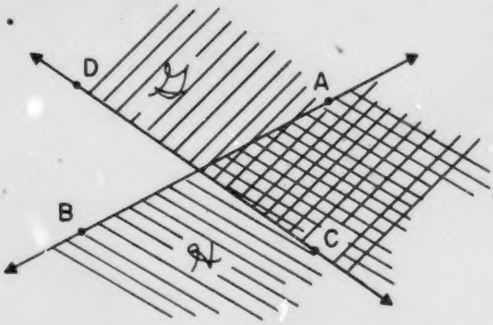
Problem Set 4-2

1. (a) By Theorem 2-10 two intersecting lines determine a unique plane. Call the unique plane determined by $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, $\mathcal{P}$. Then all the points in $\mathcal{H}$ are members of $\mathcal{P}$ and all the points of $\mathcal{G}$ are members of $\mathcal{P}$. By definition of subset, $\mathcal{H}$ and $\mathcal{G}$ are subsets of $\mathcal{P}$.

(b)

(c)

(d)



(e) cross hatched

(f) $\mathcal{H}$ is a convex set and $\mathcal{G}$ is a convex set (Plane separation Postulate) and the intersection of two convex sets is a convex set.

2. convex, halfplane, intersection, convex

3. points, $\overleftrightarrow{PQ}$, subset

4. 8, subset

5. (a) No. $\overleftrightarrow{RS}$ and $\overleftrightarrow{PB}$ intersect in, at most, one point and by hypothesis, that point is A.

(b) The empty set since A is not contained in $\overline{PB}$.

(c) No, because $\overline{PB}$ does not intersect $\overleftrightarrow{RS}$ and by the Plane Separation Postulate every segment which joins a point in one halfplane determined by a line with a point in the other halfplane determined by that line must intersect the edge of the halfplanes.

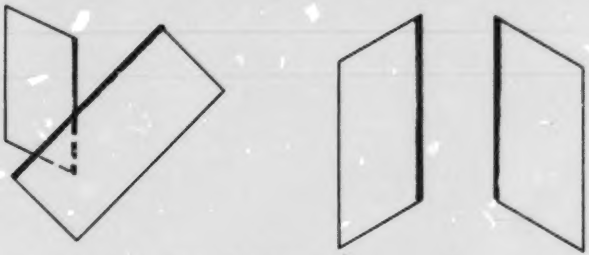
(d) Yes, since (a) and (c) have ruled out all other alternatives.

(e) Yes. By (d) P, assumed to be any point in $\overline{AB}$ except A, is on the same side of $\overleftrightarrow{RS}$ as B and therefore in the same halfplane as B.

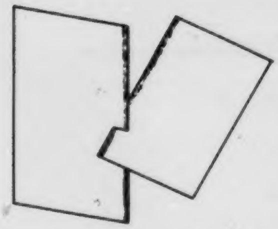
6. $\overleftrightarrow{RS}$

7. (a) Yes. Theorem 4-1 states that the intersection of any two convex sets is a convex set.

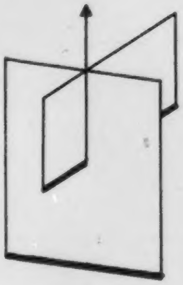
   (b) The three other types of intersections are illustrated. The heavier lines represent the edges of the halfplanes.



(1) The Empty Set



(2) The Interior of a Segment



(3) A Halfline

Problem Set 4-3

1. union, rays, endpoint, line

2. (a) No (b) No

   (c) (1) The union of two opposite rays cannot be an angle since opposite rays are collinear and the definition of an angle states that the rays which form the sides of the angle cannot lie in the same line.

   (2) The union of two rays which do not lie in the same line is an angle if and only if those rays have a common endpoint.

3. $\angle NPR$; $\angle NPT$; $\angle MPS$; $\angle MPT$

4. $\angle DEB$, $\angle BEC$, $\angle CEA$ and $\angle AED$

$\angle CEA$ could be called $\angle CEP$ and $\angle AED$ could be called $\angle PED$.

(a)	Yes	(d)	Yes
(b)	No	(e)	To all angles formed by $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$
(c)	Yes		

5.

(a)	$\angle BMR$	(d)	$\angle DNS$
(b)	$\angle RND$ or $\angle MND$	(e)	$\angle CNS$
(c)	$\angle AMN$ or $\angle AMS$	(f)	$\angle BMN$ or $\angle BMS$

Yes. $\angle RMA$ and $\angle CNM$ or $\angle CNR$

6.

(a)	60	(e)	70	(i)	70
(b)	30	(f)	15	(j)	90
(c)	30	(g)	25	(k)	125
(d)	30	(h)	70	(l)	100

Problem Set 4-4

Problem 6 is intended to help prepare for the introduction of a ray-coordinate system in a plane.

1. (a) $\frac{x}{180}$ (b) $\frac{y}{2}$ (c) $\frac{x}{200}$

2.

(a)	135	1.5	150
(b)	18	.2	20
(c)	54	.6	60
(d)	x	$\frac{x}{90}$	$\frac{10x}{9}$
(e)	$90y$	y	$100y$
(f)	117	1.3	130
(g)	$\frac{9z}{10}$	$\frac{z}{100}$	z

3. larger

4. Column I $\frac{135}{18} = 7.5$

Column II $\frac{1.5}{.2} = 7.5$

Column III $\frac{150}{20} = 7.5$

They are equal.

5. (a) $\frac{135}{54} = 2.5$ $\frac{1.5}{.6} = 2.5$ $\frac{150}{60} = 2.5$

(b) $\frac{x}{117}$, $\frac{\frac{x}{90}}{1.3} = \frac{x}{117}$, $\frac{\frac{10x}{9}}{130} = \frac{x}{117}$

*6. (a) (1) 60
(2) 75
(3) 105
(4) 90
(5) 90
(6) 135
(7) 45
(8) 150
(9) 120
(10) 60

(b) All answers given in (a) agree with the definition of angle and with Postulate 16.

(c) $p - z$, 180, 180, $360 - (p - z)$

Problem Set 4-5

1. (a) 35
(b) 85
(c) 50
(d) 45
(e) 145
(f) 95
(g) 40
(h) 90
(i) 125
(j) 130

2. (a) Tom's, Jim's, Bill's and Hank's are acceptable since they satisfy the definition for a ray-coordinate system and the Protractor Postulate. Pete's diagram places P and N on opposite sides of $\overleftrightarrow{MV}$ in violation of the Protractor Postulate.

(b) 60, 130 Definition of ray-coordinate system

(c) 70 Definition of ray-coordinate system

3. (a) (1) $y - x$ (2) $z - x$ (3) $z - y$ (4) y

(b) (1) $y - x$ or $360 - (y - x)$ whichever is less than 180
(2) $z - x$ or $360 - (z - x)$ whichever is less than 180
(3) $z - y$ or $360 - (z - y)$ whichever is less than 180
(4) y or $360 - y$ whichever is less than 180

4. (a) The union of $\overrightarrow{VM}$, $\overrightarrow{VN}$, $\overrightarrow{VP}$.
The union of $\overrightarrow{VM}$, $\overrightarrow{VN}$, $\overrightarrow{VP}$, $\overrightarrow{VQ}$

(b) (1) $\overrightarrow{VN}$ (3) V
(2) V (4) $\overrightarrow{VQ}$

(c) (1) The empty set.
(2) The set of all points in the halfplane $\mathcal{H}$ and edge $\overleftrightarrow{QM}$ of the halfplane.
(3) Two half lines; the set including all points on $\overrightarrow{VP}$ and $\overrightarrow{VN}$, except V.

5. (a) F $\angle AOB$ and $\angle BOC$ are different sets of points.
(b) T
(c) T
(d) F The inequality is in the wrong direction.
(e) T
(f) F $\angle\alpha$ and $\angle\beta$ are different sets of points.
(g) T
(h) T

6. (a) $\angle RVY$ is a set of points: 80 is a number.
(c) Sets of points cannot be added. Numbers can be added.
(d) A set of points is not a number.
(e) Sets of points cannot be added.
(g) Greater than is a relation between numbers not between a number and a set of points.
(h) Multiplication is an operation on a pair of numbers, not on a number and a set of points.
(i) In our geometry there is no angle with measure 180.
(j) In our geometry there is no angle with measure 280.

7. (a) One, by the Angle Construction Theorem.
(b) Two. By the Protractor Postulate there is a ray-coordinate system in $\mathcal{E}$ relative to A such that $\overrightarrow{AC}$ is the zero-ray. In this system the rays with ray-coordinates r, and 360 - r are the two and only two rays that form with $\overrightarrow{AC}$ an angle whose measure is r. Furthermore all rays in $\mathcal{E}$ with end-point A have been considered since all are assigned ray-coordinates in this system.

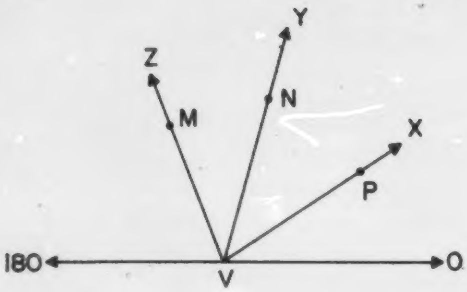
Problem Set 4-6

Problems 4 and 5 help students with the proof of Theorem 4-4 and Theorem 4-5.

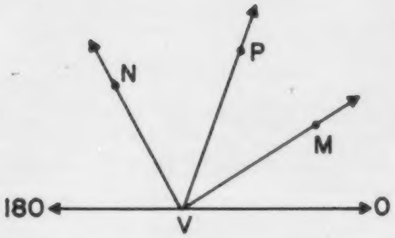
1. $\overrightarrow{VN}$; 35

2. midray, bisect

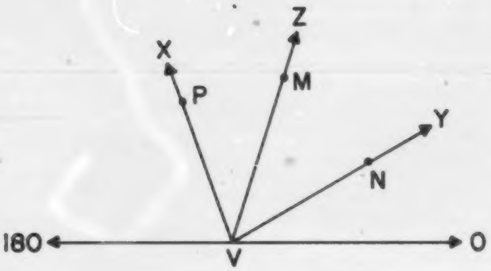
3. (a)



(b)



(c)



*4. betweenness, ray-coordinate system

$m\angle FVG = g - f$

$m\angle EVG = g - 0 = g$

Then $m\angle EVF + m\angle FVG = f + (g - f) = g$ by the additive property of equality.

It follows that $m\angle EVF + m\angle FVG = m\angle EVG$ by the transitive property of equality.

Also $m\angle EVF = m\angle EVG - m\angle FVG$ by the additive property of equality and the fact that for any number a, $a - a = 0$.

*5. 180, between, b, m$\angle$BVQ, definition, $q - 0 = q$, m$\angle$BVQ
If $q = b - q$
then $2q = b$ by the additive property of equality,
and $q = \frac{b}{2}$ by the multiplicative property of equality.
Hence, the solution is $q = \frac{b}{2}$ if there is one. It is readily checked that it is a solution.
Yes. All rays that lie between $\overrightarrow{VA}$ and $\overrightarrow{VB}$ have been considered in the argument since all have one and only one ray-coordinate in the system. Hence $\angle$AVB has only one midray.

6. (a) 3
(b) $\frac{1}{3}$
(c) $\frac{3}{7}$
(d) $\frac{1}{2}$
(e) 2
(f) 1
(g) $\frac{3}{2}$
(h) $\frac{7}{10}$

7. $\overrightarrow{VN}$ is between $\overrightarrow{VP}$ and $\overrightarrow{VM}$

8. (a) 56
(b) 128
(c) 344

9. (a) We expect that the student will use the formula in the Text at the bottom of Page 167 to find that the ray-coordinate of the midray is $\frac{x+y}{2}$. (Also see comments below.)
(b) We expect that the student will conclude from studying diagrams that $\frac{x+y}{2}$ is the opposite ray to the desired midray and therefore say that the ray-coordinate is $\frac{x+y}{2} - 180$ if $\frac{x+y}{2} \geq 180$ and that it is $\frac{x+y}{2} + 180$ if $\frac{x+y}{2} < 180$. (Also see comments below.)

A formal justification of these results can be given as follows. In this justification, we consider two concurrent rays $\overrightarrow{VA}$ and $\overrightarrow{VB}$. Suppose that in a ray-coordinate system S, the ray-coordinates of $\overrightarrow{VA}$ and $\overrightarrow{VB}$ are a and b, respectively, where $a < b$. We let $\overrightarrow{VC}$ be the ray with ray-coordinate $\frac{a+b}{2}$ in S and $\overrightarrow{VD}$ be the ray opposite to $\overrightarrow{VC}$.

We first show that the two angles, $\angle AVC$ and $\angle BVC$, that $\overrightarrow{VC}$ forms with $\overrightarrow{VA}$ and $\overrightarrow{VB}$ have the same measure. To see this, we observe that

$$0 \leq a = \frac{a + a}{2} < \frac{a + b}{2} < \frac{b + b}{2} = b < 360$$

and that since $b < 360$

$$\frac{b - a}{2} < 180 .$$

Hence,

$$m\angle AVC = \frac{a + b}{2} - a = \frac{b - a}{2}$$

and

$$m\angle BVC = b - \frac{a + b}{2} = \frac{b - a}{2}$$

We now show that the two angles, $\angle AVD$ and $\angle BVD$, that $\overrightarrow{VD}$ forms with $\overrightarrow{VA}$ and $\overrightarrow{VB}$ also have the same measure. Since $\overrightarrow{VD}$ is opposite to $\overrightarrow{VC}$, the ray-coordinate of $\overrightarrow{VD}$ is

$$\frac{a + b}{2} + 180 \quad \text{if} \quad \frac{a + b}{2} < 180$$

$$\frac{a + b}{2} - 180 \quad \text{if} \quad \frac{a + b}{2} \geq 180$$

We observe that in case $\frac{a + b}{2} < 180$, then

$$0 \leq a < b < b + 180 - \frac{b - a}{2} = \frac{a + b}{2} + 180 < 360$$

and that in case $\frac{a + b}{2} \geq 180$, then

$$0 \leq \frac{a + b}{2} - 180 = a + \frac{b - a}{2} - 180 < a < b < 360$$

since $\frac{b - a}{2} - 180$ is negative.

Consequently, in either instance,

$$m\angle AVD = 180 - \frac{b - a}{2}$$

and

$$m\angle BVD = 180 - \frac{b - a}{2}$$

To complete the justification we need to show that if $b - a < 180$, then $\overrightarrow{VC}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VB}$ and that if $b - a > 180$, then $\overrightarrow{VD}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VB}$. Let S' be the ray-coordinate system in which $\overrightarrow{VA}$ is the zero-ray and $\overrightarrow{VB}$ has its ray-coordinate less than 180.

Consider first the case that $b - a < 180$. Then, the ray-coordinate of $\overrightarrow{VB}$ in S' is $b - a$ since $m\angle AVB = b - a$. Let q be the ray-coordinate of $\overrightarrow{VC}$ in S'. Since $m\angle AVC = \frac{b - a}{2}$, we have either

$$q - 0 = \frac{b - a}{2}$$

or

$$360 - (q - 0) = \frac{b - a}{2}$$

We can see that q cannot be such that

$$360 - q = \frac{b - a}{2}, \quad \text{that is} \quad q = 360 - \frac{b - a}{2};$$

for, if it were, then $q > 180$ and computing $m\angle BVC$ in S' we obtain

$$(360 - \frac{b - a}{2}) - (b - a)$$

or

$$360 - [(360 - \frac{b - a}{2}) - (b - a)].$$

Neither of these can be $\frac{b - a}{2}$, which is $m\angle BVC$, since in the first instance this would mean $b - a = 180$ and in the second would mean $b - a = 0$. Hence q must be $\frac{b - a}{2}$. Thus, $0 < q < b - a$. Therefore, if $b - a < 180$, $\overrightarrow{VC}$ is the midray and its ray-coordinate in S is $\frac{a + b}{2}$.

Consider now the case that $b - a > 180$. Then, the ray-coordinate of $\overrightarrow{VB}$ in S' is $360 - (b - a)$ since $m\angle AVB = 360 - (b - a)$. Let q be the ray-coordinate of $\overrightarrow{VD}$ is S'. Since $m\angle AVD = 180 - \frac{b - a}{2}$, we have either that

$$q - 0 = 180 - \frac{b - a}{2}$$

or

$$360 - (q - 0) = 180 - \frac{b - a}{2}$$

We can see that q cannot be such that

$$360 - (q - 0) = 180 - \frac{b - a}{2},$$

that is $$q = 180 + \frac{b - a}{2},$$

for, if it were, then computing $m\angle BVD$ in S' we obtain

$$180 + \frac{b - a}{2} - \big(360 - (b - a)\big)$$

or

$$360 - \left[180 + \frac{b - a}{2} - \big(360 - (b - a)\big)\right].$$

Neither of these can be $180 - \frac{b - a}{2}$ which is $m\angle BVD$, since in the first instance this would mean $b - a = 180$ and in the second would mean $b - a = 360$. Hence, q must be $180 - \frac{b - a}{2} = \frac{360 - (b - a)}{2}$. Thus,

$$0 < q < 360 - (b - a) < 180.$$

Therefore, if $b - a > 180$, then $\overrightarrow{VD}$ is the midray. That is, the ray-coordinate of the midray in S is

$$\frac{a + b}{2} + 180 \quad \text{if} \quad \frac{a + b}{2} < 180,$$

$$\frac{a + b}{2} - 180 \quad \text{if} \quad \frac{a + b}{2} \geq 180,$$

in the case that $b < a$ and $b - a > 180$.

10. (a) $k = \frac{n}{p - n}$

(b) $k = \frac{p - n}{q - n}$

(c) $p = n + \frac{2}{3}(q - n)$

11. $\frac{m\angle DAP}{m\angle DAE} = h, \quad (0 < h < 1)$

$\frac{p - d}{e - d} = h$ by using the definition of a ray-coordinate system and letting p be the ray-coordinate of $\overrightarrow{AP}$.

$p - d = h(e - d)$ by using the multiplicative property of equality.

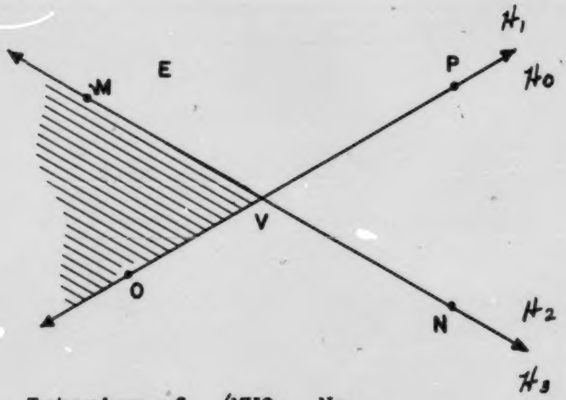
$p = d + he - hd$
or $p = (1 - h)d + he$ by using the additive property of equality and the commutative, associative and distributive properties for numbers.

Problem Set 4-7

1. (a) D, F, M
   (b) E, G, H

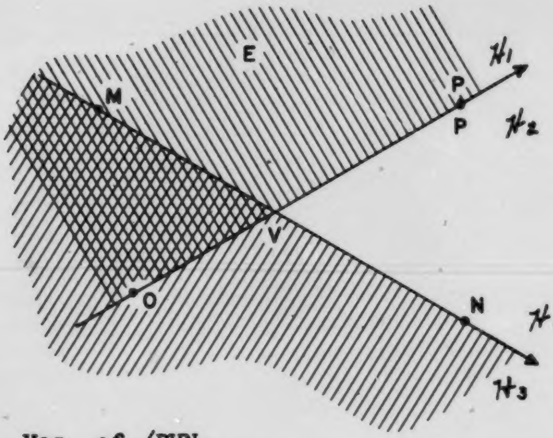
2. No. No. It lies on the angle itself.

3. (a)



(b) Yes. Interior of $\angle$MVO, No.

(c)



No, Yes--of $\angle$PVN.

4. (a) Yes. The interior of an angle is a convex set.
   (b) Not necessarily. The exterior of an angle is not a convex set.
   (c) Yes.
   (d) No. X and P are in different halfplanes with respect to at least one of the lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$. Hence $\overline{XP}$ intersects at least one of these lines, say, $\overleftrightarrow{AB}$

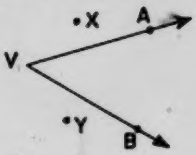
in a point Z. If Z is on $\overrightarrow{BA}$, the proof is complete; if not, X and Z are on opposite sides of $\overleftrightarrow{BC}$. Therefore $\overline{XZ}$ intersects $\overleftrightarrow{BC}$ in a point W, and X and W are on the same side of $\overleftrightarrow{AB}$. Therefore W lies on $\overrightarrow{BC}$. But W is on $\overrightarrow{XP}$. Therefore $\overrightarrow{XP}$ intersects $\angle ABC$. (Draw a figure in which X is inside the angle vertical to $\angle ABC$, and one in which X is outside that angle.)

5. (a) 70 (b) 130

6. (a) XOZ (b) YOZ (c) YOZ

7. (a) Not necessarily. $\overrightarrow{OC}$ could be between $\overrightarrow{OA}$ and $\overrightarrow{OB}$.
   (b) Yes
   (c) $\overrightarrow{OC}$ is between $\overrightarrow{OA}$ and $\overrightarrow{OB}$; the midray.

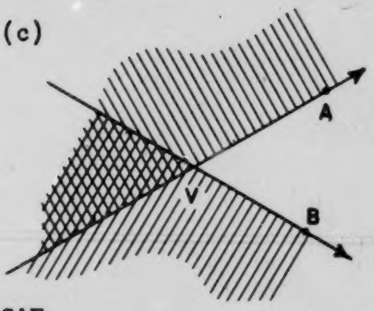
8. (a) No



Every point on $\overline{XY}$ does not lie in the exterior of $\angle AVB$.

   (b) (1) No (2) Yes

   (c)



The intersection of the two halfplanes is the cross-hatched region. This is not the exterior of $\angle AVB$. The union of the two halfplanes includes all the shaded region. This is the exterior of $\angle AVB$.

9. CAE

$m\angle BAC = m\angle DAE$ by hypothesis.

$m\angle CAD = m\angle CAD$ since they are the same number.

$m\angle BAC + m\angle CAD = m\angle DAE + m\angle CAD$ by the additive property of equality.

$m\angle BAC + m\angle CAD = m\angle BAD$ by the Betweeness-Angles Theorem.

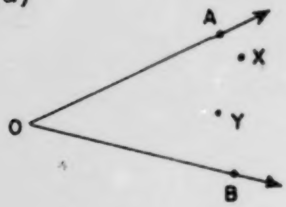
$m\angle DAE + m\angle CAD = m\angle CAE$ by the Betweeness-Angles Theorem.

$m\angle BAD = m\angle CAE$ by the substitution property of equality.

10. $m\angle CAE = m\angle DAB$ by hypothesis.
$m\angle CAD = m\angle DAC$ since they are the same number.
$m\angle CAE - m\angle CAD = m\angle DAB - m\angle DAC$ by the additive property of equality.
$m\angle CAE - m\angle CAD = m\angle DAE$ by the Betweenness-Angles Theorem.
$m\angle DAB - m\angle DAC = m\angle BAC$ by the Betweenness-Angles Theorem.
$m\angle DAE = m\angle BAC$ by the substitution property of equality.

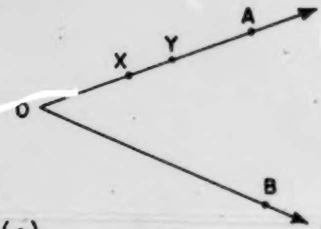
11. Let X and Y be any two points either on or in the interior of $\angle AOB$. In the following we consider the five possible positions of X and Y.

(a)



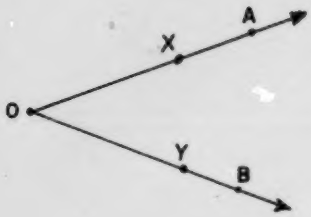
If points X and Y are both in the interior of $\angle AOB$ then $\overline{XY}$ is in the interior of $\angle AOB$ since the interior of an angle is a convex set.

(b)



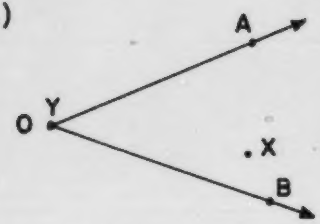
If points X and Y are both in a ray of $\angle AOB$, say $\overrightarrow{OA}$, then $\overline{XY}$ is in $\overrightarrow{OA}$ since $\overrightarrow{OA}$ is a convex set.

(c)



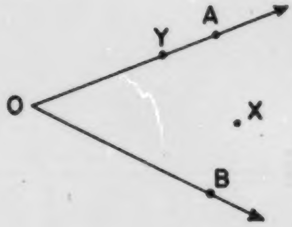
If point X is in one ray, say $\overrightarrow{OA}$, of $\angle AOB$ and Y is in the other ray of $\angle AOB$, then every point in the interior of $\overline{XY}$ is in the interior of $\angle AOB$ by the Interior of an Angle Postulate. Since X and Y are in $\angle AOB$ by hypothesis, $\overline{XY}$ is in the union of $\angle AOB$ and its interior.

(d)



Assume one point, say X, to be in the interior of ∠AOB and Y be the vertex O of ∠AOB. Then $\overrightarrow{YX}$ is between $\overrightarrow{OA}$ and $\overrightarrow{OB}$ and by the Interior of an Angle Postulate the interior of $\overrightarrow{YX}$ is in the interior of ∠AOB. Since the interior of $\overline{YX}$ is a subset of the interior of $\overrightarrow{YX}$, the interior of $\overline{YX}$ together with X is in the interior of ∠AOB. Also Y is in ∠AOB by hypothesis. Therefore every point of $\overline{YX}$ is in the union of ∠AOB and its interior.

(e)



Assume one point, say X, is in the interior of ∠AOB and that Y is in ∠AOB but is not O. Suppose Y is in $\overrightarrow{OA}$. Since Y is in $\overrightarrow{OA}$, by Theorem 4-2, the interior of $\overrightarrow{YX}$ (and hence the interior of $\overrightarrow{YX}$ and X) is contained in one of the halfplanes with edge $\overleftrightarrow{OA}$. Since X is in the interior of ∠AOB, X (and also the interior of $\overline{YX}$) is contained in the halfplane with edge $\overleftrightarrow{OA}$ which contains B. Since X is in the interior of ∠AOB, X is in the halfplane with edge $\overleftrightarrow{OB}$ which contains A. Y is in the halfplane with edge $\overleftrightarrow{OB}$ which contains A. Hence, X and Y are in the halfplane with edge $\overleftrightarrow{OB}$ containing A. Since a halfplane is a convex set, all points on $\overline{XY}$ lie in this halfplane. It follows from the Interior of an Angle Postulate that the interior of $\overline{XY}$ and X lie in the interior of ∠AOB. Since Y lies in ∠AOB by hypothesis, $\overline{XY}$ lies in the union of ∠AOB and its interior.

In all possible positions of X and Y all points in $\overline{XY}$ are in either ∠AOB or in its interior and hence are in the union of ∠AOB and its interior. Therefore the union of ∠AOB and its interior is a convex set by definition of a convex set.

Problem Set 4-8

Problem 10 helps the students develop two different proofs for Theorem 4-9. The use of this theorem shortens proof in a variety of problem situations, especially in Chapter 5. All students should consider Problem 10.

1. (a) (2)   (c) (1)   (e) (4)
   (b) (5)   (d) (6)

2. (a) They have a side in common and
   (b) the intersection of their interiors is empty.

3. (a) 25   (c) 80   (e) 55
   (b) 25   (d) 25   (f) 80

4. (a) No   (b) Yes

5. (a) linear, right
   (b) perpendicular
   (c) acute
   (d) obtuse
   (e) obtuse

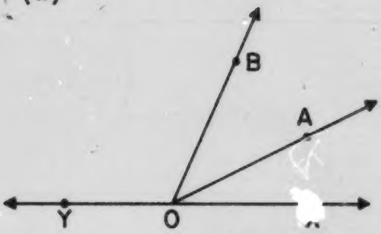
6. (a) 52
   (b) 128
   (c) 52

7. (a) 90   (e) 65
   (b) 90   (f) 155
   (c) 25   (g) 155
   (d) 65

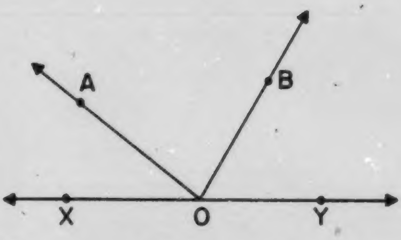
8. (b), (c), (d), (e), (f), (g)

9. (a), (c), (d), (e), (g), (h)

10. (a)



or



(b) Since $\overrightarrow{OA}$ is between $\overrightarrow{OX}$ and $\overrightarrow{OB}$ by hypothesis, the Betweenness Angle Theorem tells us that

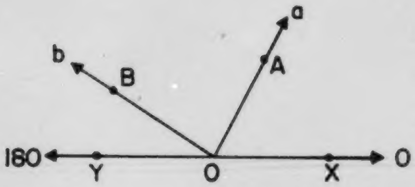
$$m\angle XOA + m\angle AOB = m\angle XOB .$$

Since $\overrightarrow{OX}$ and $\overrightarrow{OY}$ are opposite rays by hypothesis, from the definition of linear pair we know that $\angle XOB$ and $\angle BOY$ form a linear pair. It follows from Theorem 4-8 that

$$m\angle XOB + m\angle BOY = 180 .$$

Replacing $m\angle XOB$ by theequivalent sum, we have

$$m\angle XOA + m\angle AOB + m\angle BOY = 180 .$$



(c) Our hypothesis tells us that $\overrightarrow{OX}$ and $\overrightarrow{OY}$ are opposite rays and that $\overrightarrow{OX}$, $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OY}$ are coplanar in that order. Let a ray-coordinate system in the plane relative to 0 consider $\overrightarrow{OX}$ as the ray-origin. Then, by the hypothesis and the definition of a ray-coordinate system, 180 is assigned to $\overrightarrow{OY}$. By the Protractor Postulate numbers, say a and b, each less than 180 correspond to $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively. According to the given order of the rays and the definition of betweenness for rays $0 < a < b < 180$. Then by the definition of a ray-coordinate system.

$$m\angle XOA = a - 0 = a; \quad m\angle AOB = b - a; \quad m\angle BOY = 180 - b.$$

It follows, by using the addition property of equals,

$$m\angle XOA + m\angle AOB + m\angle BOY = a + (b - a) + (180 - b).$$

Simplifying gives us

$$m\angle XOA + m\angle AOB + m\angle BOY = 180 .$$

Problem Set 4-9

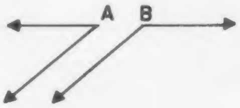
The seven starred problems in this set guide the students through the proof of Theorems 4-13, 4-14, 4-15 and 4-17 and through useful applications of these theorems. It is important that students be given the experience with proof where only a few simple relations are involved as a foundation for Chapter 5 in which proof is considered more completely and is related to more complex situations. Ask students to correct the statement of *18 in the text. The correction is given below along with the solutions to the problem.

1. (a) supplementary, supplement
   (b) complementary, complement

2. (a) 70 (c) 144 (e) $104\frac{3}{4}$ (g) n
   (b) 90 (d) 164.5 (f) 180 - n (h) 90 + n

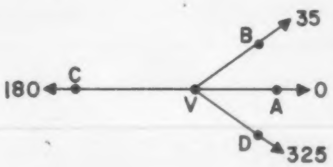
3. (a) 80 (c) 45.5 (e) 90 - x (g) x - 90
   (b) 10 (d) $52\frac{1}{2}$ (f) x (h) 45 - x

4. (a) Yes Yes
   (b) No No
   (c) (1)



If $m\angle A = 40$ and $m\angle B = 140$, $\angle A$ and $\angle B$ are supplementary but they are not adjacent and do not form a linear pair.

(2)



$m\angle CVD = 325 - 180 = 145$
$m\angle AVB = 35$
$\angle CVD$ and $\angle AVB$ are supplementary but they are not adjacent and do not form a linear pair.

5. 105, 75

$$x + (x - 30) = 180$$
$$x = 105$$
$$x - 30 = 75$$

6. 120

$$x + \frac{1}{2}x = 180$$
$$x = 120$$

7. 144, 36

$$x + \frac{1}{4}x = 180$$
$$x = 144$$
$$\frac{1}{4}x = 36$$

8. 60

$$x + \frac{1}{2}x = 90$$
$$x = 60$$

9. 72

$$180 - x = 6(90 - x)$$
$$x = 72$$

10. (a) $\overrightarrow{XR}$ and $\overrightarrow{XS}$
(b) $\angle AXS$ and $\angle BXR$

*11. 180, supplementary or adjacent

12. 90

*13. congruent, sum, supplementary, 180, right

14. 45

*15. 90, 90, acute
90, acute

16. obtuse

*17. Proof: Let $\angle a$ and $\angle b$ be congruent angles. Let $\angle c$ be any complement of $\angle a$, and let $\angle d$ be any complement of $\angle b$. We wish to show that $\angle c$ and $\angle d$ are congruent. We apply the definition of complement twice and the definition of congruent angles:

$$m\angle a + m\angle c = 90$$
$$m\angle b + m\angle d = 90$$
$$m\angle a = m\angle b$$

We conclude that

$$m\angle c = m\angle d$$

Hence $\angle c \cong \angle d$

*18. Note. Part (e) of this problem in the text should read as follows:

(e) Therefore,

$\angle AOB$ _____ $\angle COD$. Why?

(a) AOB BOC
DOC BOC
By the Betweenness-Angles Theorem.

(b) right
If two rays are $\perp$, they form a right angle.

(c) 90, 90. The measure of a right angle is 90.

(d) complement
complement
If the sum of the measures of two angles is 90, each angle is the complement of the other.

(e) $\cong$
Complements of congruent angles are congruent.

*19. (a) linear
linear

(b) supplementary
supplementary
If two angles form a linear pair, they are supplementary angles.

(c) $\angle x$, Hypothesis

(d) $\angle b \cong \angle y$
Supplements of congruent angles are congruent.

*20. (a) linear, x, linear

(b) y, r, by the Supplement Theorem

(c) $\cong$. This follows from the definition of congruent angles and from the fact that $m\angle x = m\angle x$ since every angle has a unique measure.

(d) $\angle y \cong \angle r$ since supplements of congruent angles are congruent to each other.

Problem Set 4-10

Starred Problems 4 and 5 lead toward the generalization in starred Problem 6. Problem 7 is similar to Problem 4 and Problem 8 is similar to Problem 5. It seemed desirable to include a pair of problems for group discussion and another for independent student work.

*1. (a) 90 vertical
(b) 90 linear
(c) 90 vertical

2. (a) $\angle YOB$ and $\angle XOA$
$\angle YOA$ and $\angle XOB$
$\angle YOS$ and $\angle XOR$
$\angle XOS$ and $\angle YOR$
$\angle AOS$ and $\angle BOR$
$\angle AOR$ and $\angle SOB$

(b) $\angle YOR$ and $\angle XOR$
$\angle XOR$ and $\angle XOS$
$\angle XOS$ and $\angle SOY$
$\angle SOY$ and $\angle YOR$
$\angle XOA$ and $\angle AOY$
$\angle AOY$ and $\angle YOB$
$\angle YOB$ and $\angle BOX$
$\angle BOX$ and $\angle XOA$
$\angle AOS$ and $\angle SOB$
$\angle SOB$ and $\angle BOR$
$\angle BOR$ and $\angle ROA$
$\angle ROA$ and $\angle AOS$

(c) $\angle XOA$ and $\angle AOS$
$\angle ROB$ and $\angle BOY$
$\angle XOA$ and $\angle ROB$
$\angle AOS$ and $\angle BOY$

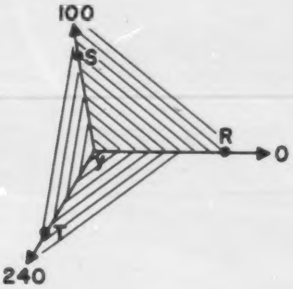
3. (a) Let $\overrightarrow{OF}$ be the ray opposite to $\overrightarrow{OE}$. $m\angle AOE = m\angle DOE = 25$, from the definition of midray. $\angle COF$ and $\angle DOE$ are vertical angles and $\angle BOF$ and $\angle AOE$ are vertical angles. Then $m\angle COF = m\angle DOE = 25$ and $m\angle BOF = m\angle AOE = 25$, from the fact that vertical angles have equal measures. Hence $\angle COF$ and $\angle BOF$ each have a measure of 25. It remains to show that $\overrightarrow{OF}$ is between $\overrightarrow{OB}$ and $\overrightarrow{OC}$. F is on the opposite side of $\overleftrightarrow{CD}$ from A, and on the opposite side of $\overleftrightarrow{AB}$ from D. Hence F is on the same side of $\overleftrightarrow{AB}$ as C, and on the same side of $\overleftrightarrow{CD}$ as B. Therefore F is interior to $\angle BOC$. Therefore ray $\overrightarrow{OF}$ is between $\overrightarrow{OB}$ and $\overrightarrow{OC}$. Hence, by definition $\overrightarrow{OF}$ is the midray of $\angle BOC$.

(b) Yes

(c) The proof would be the same as in (a) with $\frac{x}{2}$ replacing 25.

(d) The ray opposite the midray of one of a pair of vertical angles is the midray of the other vertical angle.

*4. (a)



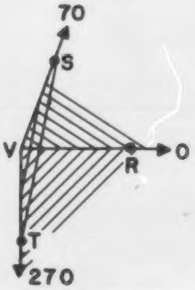
(b) The interiors of the angles are indicated by shading each at a different slant than the other two.

(c) No two of the angles have interiors which intersect.

(d) $m\angle RVS = 100$, $m\angle SVT = 140$, $m\angle TVR = 120$

(e) The sum of the measures of the three angles = 360.

*5. (a) and (b)

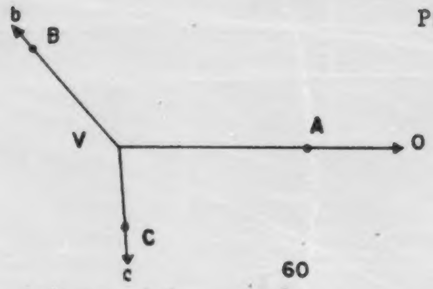


In shading the interior of $\angle SVT$, the need to have the $m\angle SVT < 180$ is considered.

(c) The interiors of $\angle SVT$ and $\angle RVS$ intersect and the interiors of $\angle SVT$ and $\angle TVR$ intersect.

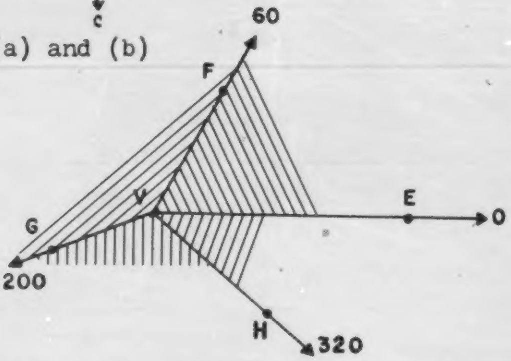
(d) $m\angle RVS = 70$, $m\angle SVT = 360 - (270 - 70) = 160$, $m\angle TVR = 90$

(e) The sum of the measures of the three angles is $m\angle RVS + m\angle SVT + m\angle TVR = 70 + 160 + 90 = 320$.

*6. 

Proof: 17, 180, hypothesis, opposite sides, greater, hypothesis, intersect, coordinate system, 360-c, (360 - c)

7. (a) and (b)

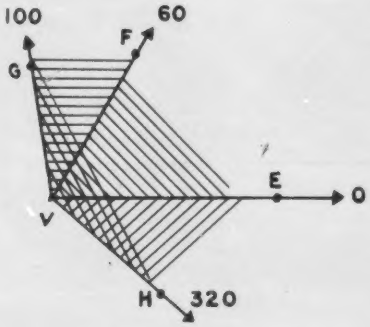


(c) No two of the angles have interiors which intersect.

(d) $m\angle EVF = 60$, $m\angle FVG = 140$, $m\angle GVH = 120$, $m\angle HVE = 40$

(e) The sum of the measures equals 360.

8. (a) and (b)



(c) The interior of $\angle GVH$ intersects with the interior of each of the angles $\angle EVF$, $\angle FVG$, $\angle HVE$.

(d) $m\angle EVF = 60 - 0 = 60$, $m\angle FVG = 100 - 60 = 40$.
$m\angle GVH = 360 - (320 - 100) = 140$, $m\angle HVE = 360 - 320 = 40$

(e) The sum of the measures $= 60 + 40 + 140 + 60 = 300$

Problem Set 4-11

1. Union, segments, noncollinear

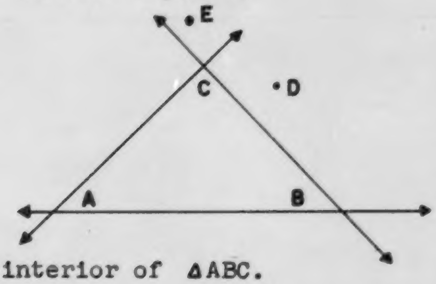
2. No. $\overline{AC}$ and $\overline{AB}$ are line segments, but the sides of $\angle A$ are rays.

3. No. Although the union contains the points of the triangle, it also contains the additional points on the rays which form the sides of the angles.

4. No. A segment joining an interior point of one side of the triangle to an interior point of another side does not lie in the triangle.

5. No.

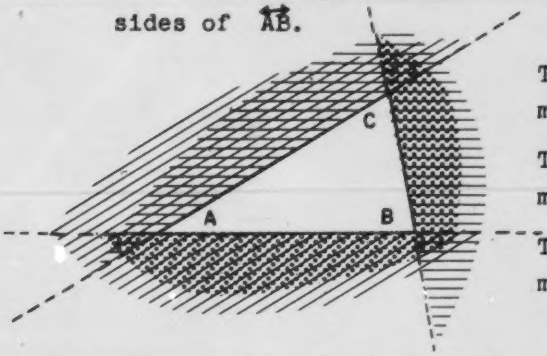
6. (a) Yes. D is such a point.
   (b) Yes. E is such a point.



7. P is in the interior of $\triangle ABC$.

8. (a) Yes
   (b) Not necessarily. P and C could be on opposite sides of $\overleftrightarrow{AB}$.

9.



The exterior of $\angle A$ is marked ▨ .

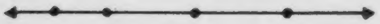
The exterior of $\angle C$ is marked ▤ .

The exterior of $\angle B$ is marked ░ .

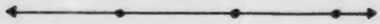
The union of the exteriors of $\angle A$, $\angle B$ and $\angle C$ is all of the shaded part.

10. c, e, h

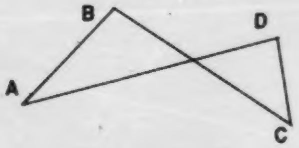
11. All points collinear



Three points collinear



$\overline{AD}$ and $\overline{BC}$ intersect at a point which is in the interior of the two segments.



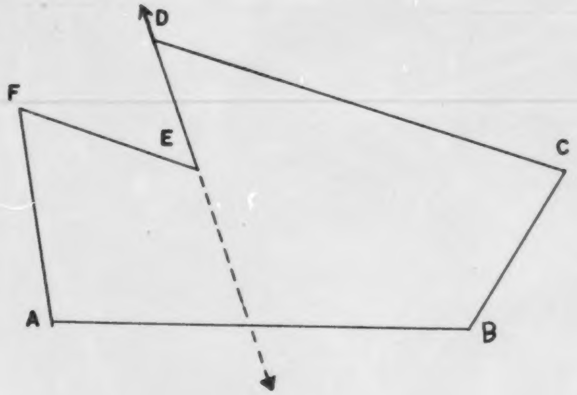
12. A and C are on opposite sides of line m.
B and C are on opposite sides of line m.
A and B are therefore, on the same side of m and in the same halfplane. Since a halfplane is convex, $\overline{AB}$ does not intersect line m.

Problem Set 4-12

1. (b) (e) (i)

2. side, diagonal, opposite, consecutive, consecutive, opposite, vertex

3. No. If $\overleftrightarrow{DE}$ is the edge of a halfplane, all of the quadrilateral except $\overline{DE}$ does not lie in the same halfplane. The same would be true if $\overleftrightarrow{EF}$ is considered the edge of a halfplane. Either of the above situations indicates that ABCDEF does not satisfy the definition of a convex polygon.



4. (b) and (c)

Polygon	Number of Sides	Number of Diagonals
Triangle	3	0
Quadrilateral	4	1
Pentagon	5	2
Octagon	8	5
Decagon	10	7

The number of diagonals in each examined case is 3 less than the number of sides. The generalization which appears correct could be stated: If n equals the number of sides of a polygon, then (n - 3) diagonals may be drawn from a single vertex.

Problem Set 4-13

1. $\angle Q - DE - P$ and $\angle P_1 - DE - Q_1$
   $\angle Q - DE - P_1$ and $\angle Q_1 - DE - P$

2. $\angle N - MP - O$
   $\angle O - NP - M$
   $\angle M - OP - N$
   $\angle P - MO - N$
   $\angle P - NO - M$
   $\angle P - MN - O$.

Chapter 4

Review Problems

1. (a) halfplane
   (b) plane
   (c) edge
   (d) convex
   (e) the empty set

2. (a) Yes
   (b) (1) Yes
   (2) No
   (3) Yes

3. (a) 80 (b) 80 (c) 105 (d) 175 (e) 105

4. (a) bisects
   (b) 0, 180, measure
   (c) acute
   (d) obtuse
   (e) congruent
   (f) right angle
   (g) congruent
   (h) acute
   (i) union, rays
   (j) triangle
   (k) segment
   (l) 65, 115
   (m) 15, 75
   (n) right angle

5. Yes, any vertex of the triangle.

6. Not necessarily. If the sum is 180 or more, it is not the measure of an angle since in our geometry all angles have measures between 0 and 180.

7. (a) 130 (b) 65 (c) 50 (d) 130
   $m\angle DAO$ is not needed.

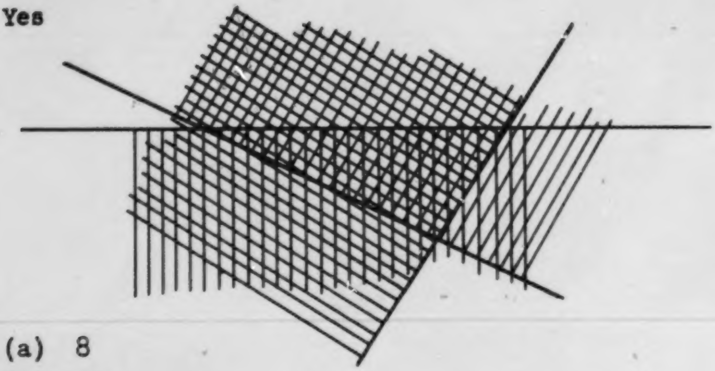
8. Yes, by Postulate 18

9. 0.7; 70

10. $b - c$ if $b > c$ and $(b - c) < 180$
    $c - b$ if $c > b$ and $(c - b) < 180$
    $360 - (b - c)$ if $b > c$ and $(b - c) > 180$
    $360 - (c - b)$ if $c > b$ and $(c - b) > 180$

11. $m\angle BAC + m\angle CAD = m\angle BAD$

12. Yes



13. (a) 8
    (b) Yes
    (c) Yes, they are the same set of points.
    (d) No

14. By the Supplement Theorem, $\angle z$ is a supplement of $\angle x$ and $\angle s$ is a supplement of $\angle y$. By the hypothesis $\angle x \cong \angle y$. Therefore, $\angle z \cong \angle s$ because supplements of congruent angles are congruent.

15. (a) 110 (b) 35 (c) 55 (d) 90

16. (a) 90

$\angle PON$ and $\angle NOQ$ form a linear pair. If $m\angle PON = x$, then $m\angle NOQ = 180 - x$.

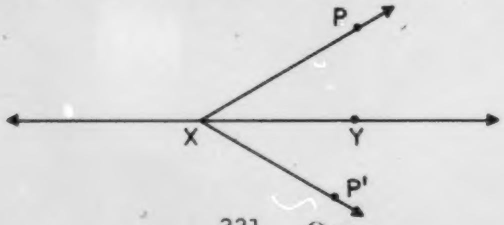
$$m\angle BON = \frac{m\angle PON}{2} = \frac{x}{2} \text{ and}$$

$$m\angle NOA = \frac{m\angle NOQ}{2} = \frac{180 - x}{2}$$

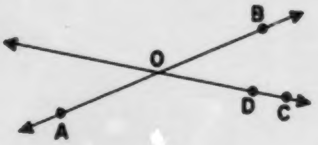
$$m\angle BON + m\angle NOA = m\angle AOB = \frac{x}{2} + \frac{180 - x}{2} = \frac{180}{2} = 90.$$

(b) Yes.

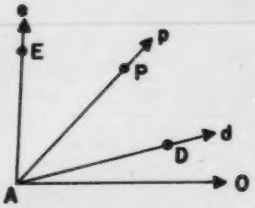
17. No. In each of the halfplanes determined by the line $\overleftrightarrow{XY}$ there is a ray $\overrightarrow{XP}$ such that $m\angle PXY = k$. There could be infinitely many such rays in space. In a given plane containing $\overrightarrow{XY}$ there are two such rays.



18. **No. Counter example: O between A and B, but not between D and C.**



19. (a)

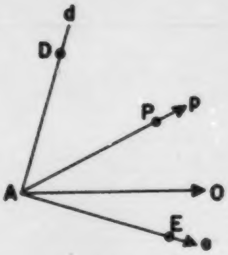


$$\frac{m\angle DAP}{m\angle EAD} = \frac{1}{2} \text{ and } 0 < e - d < 180$$

$$\frac{p - d}{e - d} = \frac{1}{2} \text{ or } p - d = \frac{1}{2}(e - d)$$

$$p = \frac{d + e}{2} .$$

(b)



$$\frac{m\angle DAP}{m\angle EAD} = \frac{1}{2} \text{ and } p < d < 90$$

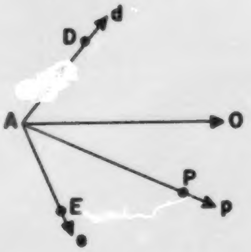
and $e > 270$

$$\frac{d - p}{360 - (e - d)} = \frac{1}{2} \text{ or}$$

$$d - p = 180 - \frac{e - d}{2}$$

From this, $p = \frac{d + e}{2} - 180.$

(c)



$$\frac{m\angle DAP}{m\angle EAD} = \frac{1}{2}; \quad d < 90; \quad 270 < e < p$$

$$\frac{360 - (p - d)}{360 - (e - d)} = \frac{1}{2}$$

$$360 - p + d = 180 - \frac{e}{2} + \frac{d}{2}$$

$$180 + \frac{d + e}{2} = p \text{ or } p = \frac{d + e}{2} + 180$$

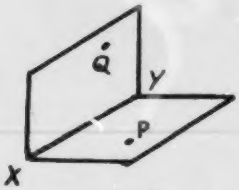
20. P is in interior of $\angle BAC$ (unless $P = A$) because P is in ray $\overrightarrow{AD}$ which is between $\overrightarrow{AC}$ and $\overrightarrow{AB}$. P may or may not be in the interior of $\triangle ABC$, depending upon whether is between A and D, or D is between A and P, or $P = D$, or $P = A$.

21. The intersection of the interiors is empty. The vertices are the same point. Adjacent angles are coplanar.

22. Convex polygon

23. Decagon

24. $\angle P - XY - Q$



25. (a) $\overline{AF}$ and E
(b) E and F
(c) $\triangle$ AFE
(d) Empty set
(e) $\triangle$ AEF

Chapter 5

ANSWERS AND SOLUTIONS

Problem Set 5-2

1. ABC ⟷ FDE
   ABC ⟷ FED
   ABC ⟷ EDF
   ABC ⟷ EFD
   ABC ⟷ DEF , a congruence
   ABC ⟷ DFE

2. MNL ⟷ PQR
   MNL ⟷ PRQ
   MNL ⟷ QPR
   MNL ⟷ QRP , a congruence
   MNL ⟷ RPQ
   MNL ⟷ RQP , a congruence

3. RST ⟷ UVW
   RST ⟷ UWV
   RST ⟷ VUW all are congruences.
   RST ⟷ VWU
   RST ⟷ WUV
   RST ⟷ WVU

4. $\triangle ABC \cong \triangle RHF$
   $\triangle LYT \cong \triangle GZD$
   $\triangle MXQ \cong \triangle LEW$
   $\triangle QPX \cong \triangle WKE$

5. $\triangle ABC \cong \triangle QPR$
   $\triangle DEF \cong \triangle FED$
   $\triangle DEF \cong \triangle SUT$
   $\triangle TUS \cong \triangle SUT$
   $\triangle FED \cong \triangle SUT$
   $\triangle KPL \cong \triangle IFJ$
   $\triangle NPO \cong \triangle GFH$

6. $\triangle ABC \cong \triangle ABC$
$\triangle ABC \cong \triangle ACB$
$\triangle ABC \cong \triangle BAC$
$\triangle ABC \cong \triangle BCA$
$\triangle ABC \cong \triangle CAB$
$\triangle ABC \cong \triangle CBA$

Problem Set 5-3a

1. (a) Reflexive property of congruence for segments
(b) Reflexive property of equality
(c) Reflexive property of congruence for angles
(d) Transitive property of congruence for triangles
(e) Transitive property of equality
(f) Addition property of equality
(g) Multiplication property of equality

2. (a) The Betweenness-Angles Theorem (Theorem 3-9).
(b) The addition property of equality
(c) Statements (a) and (b) and the substitution property of equality

3. (a) The Betweenness-Distance Theorem
(b) The addition property of equality
(c) Statements (a) and (b) and the substitution property of equality

4. (a) Definition of midpoint
(b) The multiplication property of equality
(c) Statements (a) and (b) and the substitution property of equality

5. (a) Definitions of midray and angle bisector
(b) The multiplication property of equality
(c) Statements (a) and (b) and the substitution property of equality

6. The transitive property of equality

7. The substitution property of equality or the symmetric property of equality used with the transitive property of equality

8. (a) The transitive property of congruence of angles
   (b) The transitive property of equality
   (c) The transitive property of congruence of segments
   (d) The substitution property of equality or the symmetric property of equality used with the transitive property of equality

9. Because D and F are the midpoints, respectively, of $\overline{AC}$ and $\overline{AB}$, $DC = \frac{1}{2} AC$ and $FB = \frac{1}{2} AB$ by the definition of midpoint. Then, because $AC = AB$, $\frac{1}{2} AC = \frac{1}{2} AB$ by the multiplication property of equality. It therefore follows by the substitution property of equality, that $DC = FB$, and by the definition of congruence, that $\overline{DC} \cong \overline{FB}$.

10. (a) Since $\overleftrightarrow{AB} = \overleftrightarrow{AC}$, the conclusion follows from the substitution property of equality.

    (b) The conclusion does not follow from the hypothesis. $\angle b \cong \angle c$ means $m\angle b = m\angle c$. We may, then, replace $m\angle b$ by $m\angle c$ in the hypothesis and conclude that $m\angle a + m\angle c = 90$ or that $\angle c$ is the complement of $\angle a$. This would lead to $\angle c$ being the complement of $\angle b$ only if $m\angle b = m\angle a$. Since we do not know this to be true, we can not state that $\angle c$ is the complement of $\angle b$.

11. We are asked to prove that if $\Delta ABC \cong \Delta DEF$, then $\Delta DEF \cong \Delta ABC$.
Since $\Delta ABC \cong \Delta DEF$ by hypothesis, it follows from the definition of congruence that $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $\overline{AC} \cong \overline{DF}$, $\angle A \cong \angle D$, $\angle B \cong \angle E$, $\angle C \cong \angle F$.

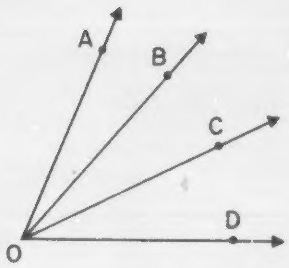
But since congruence for both segments and angles has the symmetric property, we know that
$\overline{DE} \cong \overline{AB}$, $\overline{EF} \cong \overline{BC}$, $\overline{DF} \cong \overline{AC}$, $\angle B \cong \angle A$, $\angle E \cong \angle B$,
$\angle F \cong \angle C$.

From the definition of congruence for triangles, it then follows that $\Delta DEF \cong \Delta ABC$.

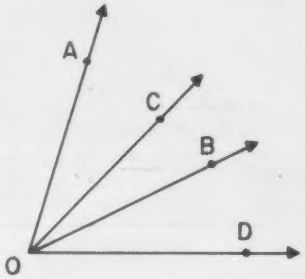
Problem Set 5-3b

1. (a) The Betweenness-Addition Theorem for points (Th. 5-4).
   (b) The Betweenness-Addition Theorem for rays (Th. 5-5).
   (c) The Betweenness-Addition theorem for points (Th. 5-4).
   (d) The Betweenness-Addition Theorem for rays (Th. 5-5).

2. Theorem 5-5 : If $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are between $\overrightarrow{OA}$ and $\overrightarrow{OD}$ and $\angle AOB \cong \angle COD$, then $\angle AOC \cong \angle BOD$.



Case 1



Case 2

(1) $m\angle AOB = m\angle COD$, because if angles are congruent, then they have equal measures.

(2) $m\angle BOC = m\angle BOC$, because of the reflexive property of equality for numbers.

(3) $m\angle AOB + m\angle BOC = m\angle COD + m\angle BOC$ (Case 1) ,
$m\angle AOB - m\angle BOC = m\angle COD - m\angle BOC$ (Case 2) ,
because of the addition property of equality.

(4) $m\angle AOB + m\angle BOC = m\angle AOC$ and
$m\angle COD + m\angle BOC = m\angle BOD$ (Case 1)
$m\angle AOB - m\angle BOC = m\angle AOC$ and
$m\angle COD - m\angle BOC = m\angle BOD$ (Case 2)
because of the Betweenness-Angles Theorem.

(5) $m\angle AOC = m\angle BOD$, because of the substitution property of equality.

(6) $\angle AOC \cong \angle BOD$, because of the definition of congruence of angles.

Problem Set 5-4

1. (a) If the measure of an angle is 90, then the angle is a right angle.
   (b) If the points of a set are coplanar, then a plane contains all the points of the set.
   (c) If the union of two rays is an angle, then these rays have a common end point and do not lie on the same line.
   (d) If $\overrightarrow{VP}$ is the midray of $\angle AVB$, then $\overrightarrow{VP}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VB}$, and $\angle AVP \cong \angle PVB$.

2. (a) If a point in a coordinate system has coordinate 1, it is the unit point of that system; if a point is the unit point of a coordinate system, it has coordinate 1 in that system.
   (b) If two distinct rays are collinear and have a common endpoint, they are opposite rays; if two rays are opposite rays, they are distinct, collinear and have a common endpoint.
   (c) If a point is the midpoint of a segment, it belongs to the segment and is equally distant from the endpoints of the segment; if a point belongs to a segment and is equally distant from the endpoints of the segment, it is the midpoint of the segment.

3. (a) An angle is an obtuse angle if and only if its measure is greater than 90

(b) A pair of angles are vertical angles if and only if their sides form two pairs of opposite rays.

(c) Two angles are a linear pair of angles if and only if they are formed by three concurrent rays, two of which are opposite rays.

4. (a) Yes

(b) No. The "then" part of the definition must agree with the statement to be justified.

(c) No. Same reason as (b)

(d) No. Same reason as (b)

(e) Yes

5. (a) If two lines are perpendicular, they form right angles.

(b) If a point is the midpoint of a segment, it belongs to the segment and is equally distant from the endpoints of the segment.

(c) If the sum of the measures of two angles is 180 , they are supplementary angles.

(d) If two angles are complementary angles, the sum of their measures is 90.

(e) If a set containing more than one point is convex, for every two points of the set, the segment joining the points is contained in the set.

(f) If two triangles are congruent, corresponding parts are congruent.

(g) If a point is in the interior of an angle, the ray whose endpoint is the vertex of the angle and which contains the given point is between the sides of the angle.

(h) If a ray is between the sides of an angle, any point in the interior of the ray lies in the interior of the angle.

Problem Set 5-5

1. In each of the following statements the hypothesis is underlined with one line and the conclusion with two lines.

   (a) If two angles are vertical angles, then they are congruent.

   (b) If angles are right angles, then they are congruent.

   (c) If angles are complements of congruent angles, then they are congruent.

   (d) If we have a line and a point not on that line, then they are contained in exactly one plane.

   (e) If a set of points is the interior of an angle, then it is a convex set.

   (f) If two sets of points are each convex sets, then their intersection is a convex set.

2. Reason 2 : The if-clause refers to Statement 1 ; the then-clause refers to Statement 2.

   Reason 3 : The if-clause refers to Statement 2 ; the then-clause refers to Statement 3.

   Reason 5 : The if-clause refers to Statement 4 ; the then-clause refers to Statement 5.

   Reason 6 : The if-clause refers to Statement 5 and the reflexive property of equality; the then-clause refers to Statement 6.

   Reason 7 : The if-clause refers to Statements 3 and 6 ; the then-clause refers to Statement 7.

   Reason 8 : The if-clause refers to Statement 7 ; the then-clause refers to Statement 8.

3. (a)

1. $m\angle A = 90$, $m\angle B = 90$.	1. Definition of right angle
2. $m\angle A = m\angle B$	2. Transitive (or substitution) property of equality
3. $\angle A \cong \angle B$	3. Definition of congruence

(b) Let $\angle a$ and $\angle b$ be congruent angles. Let $\angle\alpha$ be any supplement of $\angle a$ and let $\angle\beta$ be any supplement of $\angle b$.

1. $m\angle a = m\angle b = k$	1. Definition of congruence
2. $m\angle\alpha = 180 - m\angle a = 180 - k$. $m\angle\beta = 180 - m\angle b = 180 - k$.	2. Definition of supplementary angles
3. $m\angle\alpha = m\angle\beta$	3. Transitive (or substitution) property of equality
4. $\angle\alpha = \angle\beta$	4. Definition of congruence

(c) Let $\angle$ AWC and $\angle$ BWD be one pair of vertical angles formed when $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at W.

Statements	Reasons
1. $\angle$ AWC and $\angle$ CWB form a linear pair and $\angle$ BWD and $\angle$ CWB form a linear pair.	1. From the definition of linear pair
2. $\angle$ AWC is the supplement of $\angle$ CWB and $\angle$ BWD is the supplement of $\angle$ CWB.	2. If two angles form a linear pair, they are supplementary.
3. $\angle$ CWB $\cong$ $\angle$ CWB	3. By the reflexive property of congruence of angles.
4. $\angle$ AWC $\cong$ $\angle$ CWB	4. Supplements of congruent angles are congruent.

4. Reason 3 is one way of stating the definition of congruent segments.
Reason 4 is the transitive property or the substitution property of equality.

Reason 5 is one way of stating the definition of congruent segments.

Problem Set 5-6

1. (a) S.A.S.
(b) -----
(c) S.S.S.
(d) A.S.A.
(e) S.A.S.
(f) S.A.S.
(g) S.A.S.
(h) S.A.S.

2. (a) $\angle$ AHB
(b) $\angle$ AHB , $\angle$ ABH
(c) $\overline{BF}$
(d) $\angle$ F , $\overline{FH}$ or $\angle$ HBF , $\overline{HB}$

3. (a) $\angle$ AFB , $\angle$ B
(b) $\overline{AR}$, $\overline{RF}$
(c) $\overline{AB}$, $\overline{BR}$
(d) $\angle$ R
(e) $\overline{RF}$
(f) $\angle$ AFB

4. (a) $\overline{HB}$, $\overline{BF}$
(b) $\angle$ AHB , $\angle$ HBA
(c) $\angle$ HBF
(d) $\angle$ HBF , $\angle$ F
(e) $\angle$ A

5. (a) S.S.S
(b) Insufficient
(c) A.S.A
(d) S.S.S. or S.A.S.
(e) S.S.S.
(f) Insufficient
(g) S.A.S.
(h) A.S.A
(i) S.A.S.
(j) S.A.S.

6. (a) $\angle a \cong \angle b$ or $\overline{HF} \cong \overline{BF}$
(b) $\angle a \cong \angle b$ or $\overline{HF} \cong \overline{BF}$
(c) A.S.A
(d) $\angle A = \angle M$ or $\overline{QR} \cong \overline{WR}$
(e) $\overline{AR} \cong \overline{MR}$
(f) $\overline{XF} \cong \overline{KF}$ or $\angle XYF \cong \angle KYF$
(g) $\overline{XY} \cong \overline{KY}$ or $\angle XFY \cong \angle KFY$

Problem Set 5-7a

In this set of problems the proofs, except for a few steps in Problems 9 , 14 and 15 , are written to comply strictly with the wording of the theorems, postulates or definitions used. Most teachers believe that students become more alert to phrasings and meanings of statements if they are held to this type of sequence in their early work with proofs. Later, when a class displays a grasp of the significance of this procedure, it may seem desirable (1) to permit students to combine two or more statements into one, (2) to consider that a statement given in one form also embodies another derived form, or (3) even to omit statements.

Obviously if permitted to write proofs in abbreviated form, students can consider more problems and have more experience with the thinking involved. Students need to be given a clear understanding as to what type of combining will be accepted and at what time in the study it may being. Occasional returning to the unabbreviated form keeps them aware of the demands of a complete proof.

As examples of shortening proofs consider the following which some teachers regard as acceptable after the students have demonstrated an understanding of the more rigorous form.

1. If the hypothesis states the equality of the measures of two parts, a student might use either the statement of the equality of measure or a statement of the congruences of these parts and give "hypothesis" for the reason. Thus for example we would omit steps 1 and 3 in Problem 5. As an intermediate step, a student might put the congruence and equality steps together as in Problems 14 and 15 below.

2. After stating a congruence between two triangles, a student might go directly to the statement of the equality of measure of corresponding parts without first stating the congruence of those parts as the precise use of the definition of congruence of triangles would necessitate. Thus in Problem 5 below, we would omit statement 7 and use the present statement 8 justifying it with the present reason 7.

3. A more drastic cutting might permit omitting statements such as steps 2 and 3 in problem 10 below or step 2 in Problem 13.

4. Sometimes no single theorem, definition or postulate includes the exact statement needed for a proof yet the needed meaning is implied by one or more accepted statements. As an example if a line $\overleftrightarrow{AB}$ is given to bisect $\overleftrightarrow{XY}$ at R, it is awkward to combine the precise definitions of midpoint of a segment and bisector of a segment to justify the statement $XR = YR$. A more general reason "the meaning of bisect" could be considered acceptable.

In connection with the proof for Problem Set 5-7b further variation in the form of proofs will be considered.

In the solutions given for Problem Set 5-8 proofs are shortened as indicated at the beginning of the set.

In later sets we give only brief suggestions for the proofs leaving the details to be filled in by each teacher according to the needs of the class.

Problem 16 in this set is starred since it gives the first experience with overlapping triangles which are considered in the next section.

1.
   2. Hypothesis
   3. $\angle ACB$, $\angle DCE$ are vertical angles. Hypothesis
   4. Angles, then they are congruent.
   5. S.A.S Postulate (1, 2, 4)
   6. Congruent (5), the corresponding parts are congruent.

2.
   2. Two angles are right angles (1), then they are congruent.
   4. $\overline{RO} \cong \overline{SO}$. By the definition of bisect (3), and congruence of segments.
   5. vertical angles; (3)
   6. vertical; (5)
   7. A.S.A. Postulate (2, 4, 6)
   8. $\overline{RT} \cong \overline{SQ}$. If two triangles are congruent (7), the corresponding parts are congruent.

3. 1 Hypothesis
3. They form a right angle. (2)
4. Hypothesis
5. Same as 3
6. congruent (3 , 5)
7. Reflexive property for congruence
8. S.A.S. Postulate (1, 6, 7)
9. If two triangles are congruent (8), the corresponding parts are congruent.

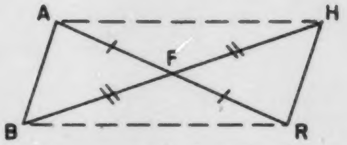
4.

(a)	1. AD = BC	1. Hypothesis
	2. $\overline{AD} \cong \overline{BC}$	2. Segments of the same measure (1) are congruent.
	3. ∠ DAR and ∠ CBR are right angles.	3. Hypothesis
	4. ∠ DAR ≅ ∠ CBR	4. Right angles are congruent
	5. AR = RB	5. Definition of midpoint
	6. $\overline{AR} \cong \overline{BR}$	6. Segments having the same measure are congruent.
	7. △ DAR = △ CBR	7. S.A.S. Postulate(2,4,6)
	8. $\overline{RC} \cong \overline{RD}$	8. Definition of congruence for triangles (7)
(b)	1. ∠ ADR = ∠ BCR ∠ DRA = ∠ CRB	1. Step 7 above and definition of congruence for triangles
	2. ∠ RDC is the complement of ∠ ADR. ∠ RCD is the complement of ∠ BCR.	2. Definition of complementary angles
	3. ∠ RDC ≅ ∠ RCD	3. Complements of congruent angles are congruent.

5.	1. $AB = FH$	1. Hypothesis
	2. $\overline{AB} \cong \overline{FH}$	2. Segments having equal measure (1) are congruent
	3. $m\angle x = m\angle y$	3. Hypothesis
	4. $\angle x \cong \angle y$	4. Angles having the same measure (3) are congruent
	5. $\overline{BH} \cong \overline{HB}$	5. Reflexive property of congruence
	6. $\triangle ABH \cong \triangle FHB$	6. S.A.S., Postulate (2 , 4 , 5)
	7. $\angle A \cong \angle F$	7. Definition of congruence for triangles (6)
	8. $m\angle A = m\angle F$	8. Congruent angles have the same measures. (7)
6.	1. $RS = TS$ and $UR = UT$	1. Hypothesis
	2. $\overline{RS} \cong \overline{TS}$, and $\overline{UR} \cong \overline{UT}$	2. Segments having the same measure (1) are congruent
	3. $\overline{SU} \cong \overline{SU}$	3. Reflexive property of congruence
	4. $\triangle RSU \cong \triangle TSU$	4. S.S.S. Postulate (2 , 3)
	5. $\angle STU \cong \angle SRU$	5. Definition of congruence for triangles
7	1. $m\angle ABH = m\angle FBH$	1. Hypothesis
	2. $\angle ABH \cong \angle FBH$	2. Angles having the same measure (1) are congruent
	3. $\angle x \cong \angle y$	3. Hypothesis
	4. $\overline{HB} \cong \overline{HB}$	4. Reflexive property of congruence
	5. $\triangle ABH \cong \triangle FBH$	5. A.S.A., Postulate (2 , 3 , 4)
	6. $\overline{AH} \cong \overline{FH}$	6. Definition of congruence for triangles (5)
	7. $AH = FH$	7. Congruent segments have the same measure. (6)

8. 1. $\angle$ BFA and $\angle$ DEC are right angles.	1. Hypothesis
2. $\angle BFA \cong \angle DEC$	2. Right angles are congruent
3. $m\angle x = m\angle y$	3. Hypothesis
4. $\angle x \cong \angle y$	4. Angles of the same measure (3) are congruent.
5. $BF = DE$	5. Hypothesis
6. $\overline{BF} \cong \overline{DE}$ (s)	6. Segments having the same measure (5) are congruent.
7. $\triangle BFA \cong \triangle DEC$	7. A.S.A. Postulate (2, 4, 6)
8. $\overline{FA} \cong \overline{EC}$	8. Definition of congruence for triangles (7)
9. $FA = EC$	9. Congruent segments have equal measure. (8).

9. Hypothesis: $\overline{AR}$ and $\overline{BH}$ bisect each other at F.
Prove: $\overline{AH} \cong \overline{BR}$



1. $\overline{AF} \cong \overline{RF}$ $\overline{BF} \cong \overline{HF}$	1. Definition of bisect
2. $\angle AFH \cong \angle RFB$	2. Vertical angles are congruent.
3. $\triangle AFH \cong \triangle RFB$	3. S.A.S. Postulate (1 , 2)
4. $\overline{AH} \cong \overline{RB}$	4. Definition of congruence for triangles (3)

10. 1. $\angle x \cong \angle r$	1. Hypothesis
2. $\angle x$ and $\angle$ HFA form a linear pair.	2. Definition of linear pair
3. $\angle r$ and $\angle$ BFA form a linear pair.	3. Definition of linear pair
4. $\angle$ HFA is a supplement of $\angle x$.	4. Angles which form a linear pair are supplementary. (2)
5. $\angle$ BFA is a supplement of $\angle r$.	5. Angles which form a linear pair are supplementary. (2)
6. $\angle HFA \cong \angle BFA$	6. Supplements of congruent angles are congruent. (4)

	7.	$m\angle HAF = m\angle BAF$		7. Definition of angle bisector
	8.	$\angle HAF \cong \angle BAF$	(A)	8. Angles having the same measure (7) are congruent.
	9.	$\overline{AF} \cong \overline{AF}$	(S)	9. Reflexive property of congruence
	10.	$\Delta HFA \cong \Delta BFA$		10. A.S.A., Postulate (6 , 7 , 8)
	11.	$\overline{FH} \cong \overline{FB}$		11. Definition of congruence for triangles (10)
	12.	$FH = FB$		12. Congruent segments have the same measure.(11)
11.	1.	$\overline{DF} \cong \overline{AC}$ $\overline{FE} \cong \overline{CB}$ $\overline{DE} \cong \overline{AB}$		1. Hypothesis
	2.	$\Delta DFE \cong \Delta ACB$		2. S.S.S. Postulate (1)
	3.	$\angle DFE \cong \angle ACB$		3. Definition of congruence for triangles (2)
	4.	$m\angle DFE = m\angle ACB$		4. Congruent angles have the same measure. (3)
12.	1.	$BD = BF$ and $ED = EF$		1. Hypothesis
	2.	$\overline{BD} \cong \overline{BF}$ and $\overline{ED} \cong \overline{EF}$		2. Segments having the same measure are congruent (1)
	3.	$\overline{BE} \cong \overline{BE}$		3. Reflexive property of congruence.
	4.	$\Delta BDE \cong \Delta BFE$		4. S.S.S. Postulate (2 , 3)
	5.	$\angle DBE \cong \angle FBE$		5. Definition of congruence for triangles (4)
	6.	$m\angle DBE = ,\angle FBE$		6. Congruent segments have the same measure. (5)
	7.	$\overrightarrow{BE}$ is the midray of $\angle ABC$ and $\overrightarrow{BE}$ bisects $\angle ABC$		7. Definition of midray and of angle bisector (6)

13.	1. $\angle A \cong \angle B$	1. Hypothesis
	2. F is between A and D Q is between B and C	2. Hypothesis
	3. $AF = AD - FD$ $BQ = BC - QC$	3. Betweenness-Distance Theorem
	4. $\overline{DF} \cong \overline{CQ}$	4. Hypothesis
	5. $FD = QC$	5. Definition of congruence for segments (4)
	6. $AD = BC$	6. Hypothesis
	7. $AD - FD = BC - QC$	7. Addition property of equality (5 , 6)
	8. $AF = BQ$	8. Substitution property of equality (3)
	9. $\overline{AF} \cong \overline{BQ}$	9. Definition of congruence for segments (8)
	10. R is the midpoint of $\overline{AB}$	10. Hypothesis
	11. $AR = RB$	11. Definition of midpoint (10)
	12. $\overline{AR} \cong \overline{RB}$	12. Definition of congruence for segments (11)
	13. $\triangle FAR \cong \triangle QBR$	13. S.A.S. Postulate (1 , 9 , 12)
	14. $\overline{RF} \cong \overline{RQ}$	14. Corresponding parts of congruent triangles are congruent. (13)
14.	1. E and F are midpoints of $\overline{AD}$ and $\overline{BC}$, respectively.	1. Hypothesis
	2. $ED = \frac{1}{2} AD$ $FC = \frac{1}{2} BC$	2. Definition of midpoint (2)
	3. $AD = BC$ ($\overline{AD} \cong \overline{BC}$)	3. Hypothesis
	4. $\overline{ED} \cong \overline{FC}$ ($ED = FC$)	4. Multiplication property of equality (2 , 3)

5. $\angle EDG \cong \angle FCH$	5. Hypothesis
6. D , G , H , C collinear in that order and $\overline{CG} \cong \overline{DH}$	6. Hypothesis
7. $\overline{DG} \cong \overline{CH}$	7. Betweenness-Addition Theorem (6)
8. $\triangle EDG \cong \triangle FCH$	8. S.A.S. Postulate (4 , 5 , 6)
9. $\overline{EG} \cong \overline{FH}$	9. Corresponding parts of congruent triangles are congruent. (8)

15.	1. E and F are midpoints of $\overline{AB}$ and $\overline{AD}$ respectively $\overline{AE} \cong \overline{AF}$ or $AE = AF$	1. Hypothesis
	2. $AB = 2AE$, $AD = 2AF$	2. Definition of midpoint (1)
	3. $AB = AD$ or $\overline{AB} \cong \overline{AD}$	3. Multiplication property of equality (1 , 2)
	4. $\angle EAC \cong \angle FAC$	4. Hypothesis
	5. $\overline{AC} \cong \overline{AC}$	5. Reflexive property of congruence
	6. $\triangle BAC \cong \triangle DAC$	6. S.A.S. Postulate (3 , 4 , 5).

*16.	1. $\angle SPR \cong \angle QRP$ $\overline{SP} \cong \overline{QR}$	1. Hypothesis
	2. $\overline{PR} \cong \overline{RP}$	2. Reflexive property of congruence
	3. $\triangle SPR \cong \triangle QRP$	3. S.A.S. Postulate (1 , 2)
	4. $\overline{SR} \cong \overline{QP}$	4. Corresponding parts of congruent triangles are congruent.

Problem Set 5-7b

This problem set offers opportunity for considerable variation in the form of proof expected from the students. The proof given here for Problems 1 through 5 ignores the Betweenness-Addition Theorem for Points (Theorem 5-4) and the Betweenness-Addition Theorem for Rays (Theorem 5-5). Many teachers will prefer this (perhaps even for the other problems in the set) since those theorems were introduced in Section 5-3, with what some may consider insufficient build-up to impress students with their significance.

The use of Theorems 5-4 and 5-5 would modify the proofs as follows:

Problem 1(a) : Steps 7, 8, 9, 10 would be omitted. Part (a) would begin with step 6, Step 7 would state that S, K, L, R are collinear. Reason--Hypothesis. Statement 11 would become statement 8. Reason 8 would be the Betweenness-Addition Theorem for Points.

Problem 2 : Steps 2, 3, 4, 5, 6 would be omitted. Statement 2 would be $\overrightarrow{RS}$ and $\overrightarrow{RT}$ are between $\overrightarrow{RX}$ and $\overrightarrow{RW}$ with Reason 2 as Hypothesis. Statement 7 would become statement 3. Reason 3 would be the Betweenness-Addition Theorem for Rays.

Problem 3 : Steps 4, 5, 6, 7, 8 would be omitted. Statement 4 would be $\overrightarrow{RX}$ and $\overrightarrow{RB}$ are between $\overrightarrow{RA}$ and $\overrightarrow{RY}$. Reason 4 would be Hypothesis. Statement 9 would become statement 5. Reason 5 would be the Betweenness-Addition Theorem for Rays.

Further shortening as mentioned before Problem Set 7-a could in Problem 2 combine steps 1 and 2 into a single step and in Problem 3 combine present steps 10 and 11 into one step and go directly from present step 12 to step 14 with the definition of congruence for triangles as the justification.

The proof for Problem 5 is given in both the long form and in one version of a shortened form.

Beginning with Problem 6 partially shortened forms of proof are used, to be modified as best fits the student's need.

Problem 11 is much more complicated than a hasty reading might suggest.

1. In $\triangle TKS$ and $\triangle TLR$

	1. $\angle x \cong \angle y$	1. Hypothesis
	2. $\angle u \cong \angle v$	2. Hypothesis
	3. $TK = TL$	3. Hypothesis
	4. $\overline{TK} \cong \overline{TL}$	4. Segments of the same measure are congruent.(3)
	5. $\triangle TKS \cong \triangle TLR$	5. A.S.A. Postulate (1 , 2 , 4)
	6. $\overline{SK} \cong \overline{RL}$	6. Definition of congruence for triangles (5)
(a)	7. $SK = RL$	7. Congruent segments have the same measure. (6)
	8. $SK + KL = RL + KL$	8. Addition property of equality (7)
	9. $SK + KL = SL$ $RL + LK = RK$	9. The Betweenness-Distance Theorem
	10. $SL = RK$	10. Substitution property of equality (8 , 9)
	11. $\overline{SL} = \overline{RK}$	11. Segments with the same measure are congruent.(10)

	12. $\overline{ST} \cong \overline{RT}$	12. Definition of congruence for triangles (5)
(b)	13. $\Delta STL \cong \Delta RTK$	13. S.S.S. Postulate (4 , 11 , 12)
2.	1. $\angle SRX \cong \angle TRW$	1. Hypothesis
	2. $m\angle SRX = m\angle TRW$	2. Congruent angles have the same measure (1)
	3. $m\angle SRX + m\angle SRT = m\angle TRW + M\angle SRT$	3. Addition property of equality (2)
	4. $m\angle SRX + m\angle SRT = m\angle XRT$	4. Betweenness-Angles Theorem
	5. $m\angle TRW + m\angle SRT = m\angle WRS$	5. Betweenness-Angles Theorem
	6. $m\angle WRS = m\angle XRT$	6. Substitution property of equality (3 , 4 , 5)
	7. $\angle WRS = \angle XRT$	7. Angles of the same measure are congruent. (6)
	8. $\overline{RS} \cong \overline{RT}$	8. Hypothesis
	9. $\overline{RW} \cong \overline{RX}$	9. Hypothesis
	10. $\Delta WRS \cong \Delta XRT$	10. S.A.S. Postulate (7 , 8 , 9)
	11. $\angle X \cong \angle W$	11. Definition of congruence for triangles
3.	1. $\overline{AR} \perp \overline{RX}$, and $\overline{BR} \perp \overline{RY}$	1. Hypothesis
	2. $\angle BRY$ is a right angle and $\angle ARX$ is a right angle.	2. Definition of perpendicular lines (1)
	3. $\angle BRY \cong \angle ARX$	3. Right angles are congruent (2)
	4. $m\angle BRY = m\angle ARX$	4. Congruent angles have equal measures. (3)
	5. $m\angle BRY + m\angle BRX = m\angle ARX + m\angle BRX$	5. Addition property of equality

6.	$m\angle BRY + m\angle BRX = m\angle XRY$	6.	Betweenness-Angles Theorem
7.	$m\angle ARX + m\angle BRX = m\angle ARB$	7.	Betweenness-Angles Theorem
8.	$m\angle XRY = m\angle ARB$	8.	Substitution property of equality
9.	$\angle XRY \cong \angle ARB$	9.	Angles of the same measure are congruent.
10.	$AR = RX$, and $BR = RY$	10.	Hypothesis
11.	$\overline{AR} \cong \overline{RX}$ and $\overline{BR} \cong \overline{RY}$	11.	Segments of the same measure are congruent.
12.	$\Delta ARB \cong \Delta XRY$	12.	S.A.S.
13.	$\overline{AB} \cong \overline{XY}$	13.	Definition of congruence for triangles
14.	$AB = XY$	14.	Congruent segments have the same measure.

---

4. Assume P , T , Q collinear and Q , S , R collinear.

1.	$\angle OPQ \cong \angle QRO$	1.	Hypothesis
2.	$m\angle OPQ = m\angle QRO$	2.	Congruent angles have equal measures.
3.	$\angle x \cong \angle y$	3.	Hypothesis
4.	$m\angle x = m\angle y$	4.	Congruent angles have equal measures.
5.	$m\angle OPQ - m\angle x = \angle QRO - m\angle y$	5.	Addition property of equality (3 , 4)
6.	$m\angle QPR = m\angle OPQ - m\angle x$, $m\angle ORP = m\angle QRO - m\angle y$	6.	Betweenness-Angles Theorem
7.	$m\angle QPR = m\angle ORP$	7.	Substitution property of equality (5 , 6)
8.	$\angle QPR \cong \angle ORP$	8.	Angles of the same measure are congruent.(7)
9.	$\overline{ST}$ bisects $\overline{PR}$ at K.	9.	Hypothesis
10.	$\overline{PK} \cong \overline{RK}$	10.	Meaning of bisect for angles (9)

11.	$\angle SKR \cong \angle TKP$	11.	Vertical angles are congruent.
12.	Δ SKR $\cong$ Δ TKP	12.	A.S.A. Postulate (8 , 10 , 11)
13.	$\overline{RS} \cong \overline{PT}$	13.	Definition of congruence
14.	RS = PT	14.	Congruent segments have equal measures.

5. Long Form

1.	$\angle A \cong \angle B$	1.	Hypothesis
2.	$\angle x \cong \angle y$	2.	Hypothesis
3.	$\overline{AG} \cong \overline{FB}$	3.	Hypothesis
4.	AG = FB	4.	Definition of congruence for segments (3)
5.	AG + GF = FB + GF	5.	Addition property of equality (4)
6.	AG + GF = AF FB + GF = BG	6.	Betweenness Distance Theorem
7.	AF = BG	7.	Substitution property of equality (5 , 6)
8.	$\overline{AF} = \overline{BG}$	8.	Definition of congruence for segments (7)
9.	Δ ADF = Δ BEG	9.	A.S.A. Postulate (1 , 2 , 8)
10.	$\overline{DF} \cong \overline{EG}$	10.	Corresponding parts of congruent triangles are congruent. (9)

5. Shortened Form

In Δ AFD and Δ BGE

1.	$\angle A \cong \angle B$ $\angle x \cong \angle y$	1.	Hypothesis
2.	$\overline{AG} \cong \overline{BF}$ A , G , F , B are collinear.	2.	Hypothesis

	3. $\overline{AF} \cong \overline{BG}$	3. Betweenness-Addition Theorem for Points (1 , 2)
	4. $\Delta AFD \cong \Delta BGE$	4. A.S.A. Postulate (1 , 3)
	5. $\overline{DF} \cong \overline{EG}$	5. Corresponding parts of congruent triangles are congruent. (4)

6.	1. $AH = BF$ $XH = YF$	1. Hypothesis
	2. $AH - XH = BF - YF$	2. Additive property of equality (1)
	3. X is between A and H , Y is between B and F.	3. Hypothesis
	4. $AH - XH = AX$ $BF - YF = BY$	4. Betweenness-Distance Theorem (3)
	5. $AX = BY$	5. Substitution property of equality (4)
	6. $\overline{AX} = \overline{BY}$	6. Definition of congruence for segments (5)
	7. $\angle A \cong \angle B$	7. Hypothesis
	8. $\overline{AB} \cong \overline{BA}$	8. Reflexive property of congruence
	9. $\Delta AXB \cong \Delta BYA$	9. S.A.S. Postulate (6 , 7 , 8)
	10. $\overline{AY} \cong \overline{BX}$	10. Corresponding parts of congruent triangles are congruent. (9)

7.	1. $\angle g \cong \angle h$	1. Hypothesis
	2. $\overrightarrow{SV}$ and $\overrightarrow{SU}$ are between $\overrightarrow{SR}$ and $\overrightarrow{ST}$	
	2. $\angle RSU \cong \angle TSV$	2. Betweenness-Addition Theorem for Rays (1)
	3. $RS = TS$ or $\overline{RS} \cong \overline{TS}$	3. Definitions of midpoint and congruence of segments

4. SU = SV or SU $\cong$ SV	4. Hypothesis and definition of congruence of segments
5. $\triangle$ RSU $\cong$ $\triangle$ TSV	5. S.A.S. Postulate (2 , 3 , 4)
6. $\angle U \cong \angle V$	6. Corresponding parts of congruent triangles are congruent. (5)

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8. In $\triangle$ BGA and $\triangle$ CFA

1. BG = CF or $\overline{BG} \cong \overline{CF}$ BA = CA or $\overline{BA} \cong \overline{CA}$	1. Hypothesis and definition of congruence for segments
2. A is the midpoint of $\overline{GF}$.	2. Hypothesis
3. AG = AF or $\overline{AG} \cong \overline{AF}$	3. Definitions of midpoint and congruence of segments (2)
4. $\triangle$ BGA = $\triangle$ CFA	4. S.S.S. Postulate (1 , 2 , 3)
5. $\angle BGA \cong \angle CFA$	5. Corresponding parts of congruent triangles are congruent. (4)
In $\triangle$ BGF and $\triangle$ CFG	
6. $\overline{GF} \cong \overline{FG}$	6. Reflexive property of congruence
7. $\triangle$ BGF $\cong$ $\triangle$ CFG	7. S.A.S. Postulate (1 , 4 , 5)
8. $\overline{BF} \cong \overline{CG}$	8. Corresponding parts of congruent triangles are congruent. (7)

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9. In $\triangle$ AED and $\triangle$ AEB

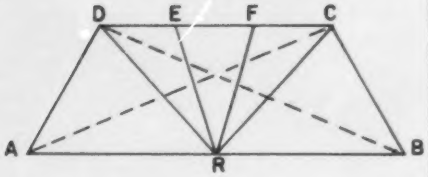
1. AD = AB or $\overline{AD} \cong \overline{AB}$ ED = EB or $\overline{ED} \cong \overline{EB}$	1. Hypothesis and definition of congruence of segments
2. $\overline{AE} \cong \overline{AE}$	2. Reflexive property of congruence
3. $\triangle$ AED $\cong$ $\triangle$ AEB	3. S.S.S. Postulate (1 , 2)

4. $\angle B \cong \angle ADE$	4. Corresponding parts of congruent triangles are congruent. (3)
5. $m\angle B = 90$	5. Hypothesis and definition of right angle
6. $m\angle ADE = 90$	6. Congruent angles have the same measure. (4 , 5)
7. $\angle ADE$ is a right angle.	7. Definition of right angle (6)
8. $\angle DAE \cong \angle BAE$	8. Corresponding parts of congruent triangles are congruent. (3)
9. $\overrightarrow{AE}$ bisects $\angle DAB$ or $\overrightarrow{AE}$ bisects $\angle A$	9. Definition of angle bisector (8)

10. In $\Delta\, KLJ$ and $\Delta\, MLP$

1. $LK = LM$ or $\overline{LK} \cong \overline{LM}$ $LJ = LP$ or $\overline{LJ} \cong \overline{LP}$	1. Hypothesis, definition of bisect and definition of congruence of segments
2. $\angle KLJ$ and $\angle MLP$ are vertical angles.	2. Definition of vertical angles
3. $\angle KLF \cong \angle MLP$	3. Vertical angles are congruent. (2)
4. $\Delta\, KLJ \cong \Delta\, MLP$	4. S.A.S. Postulate (1 , 3)
5. $\overline{JK} \cong \overline{PM}$ and $\angle J \cong \angle P$	5. Corresponding parts of congruent triangles are congruent. (4)
6. $\angle x \cong \angle y$	6. Hypothesis
7. $\Delta\, JKQ \cong \Delta\, PMR$	7. A.S.A. Postulate (5 , 6)
8. $\overline{KQ} \cong \overline{MR}$	8. Corresponding parts of congruent triangles are congruent. (7)

11.



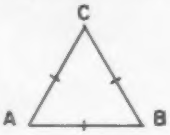
A proof is suggested but not completed.

1. Prove $\triangle ADB \cong \triangle BCA$ Then $\overline{DB} \cong \overline{CA}$	1. S.A.S.
2. Prove $\triangle ADC \cong \triangle BCD$ Then $\angle ADC \cong \angle BCD$	2. S.S.S.
3. Prove $\triangle ADR \cong \triangle BCR$ Then $\overline{DR} \cong \overline{CR}$ and $\angle ADR \cong \angle BCR$	3. S.A.S.
4. $\angle RDE \cong \angle RCF$	4. Properties of equality and the Betweenness-Angle Theorem
5. $\overline{DE} \cong \overline{CF}$	5. Betweenness-Addition Theorem for Points
6. $\triangle DER \cong \triangle CFR$	6. S.A.S.
7. $RE = RF$	7. Definition of congruence for triangles

Problem Set 5-8

In Problem 5(a), (step 1), Problem 6, (step 5) and Problem 8 (step 1) combining has been done. Note that the reason for step 5 in Problem 6 is appropriate for the changed form.

1. In $\triangle ABC$, $\overline{AB} \cong \overline{BC} \cong \overline{AC}$
   Prove: $\angle C \cong \angle A \cong \angle B$



Statements	Reasons
1. $\overline{AB} \cong \overline{BC}$	1. Hypothesis
2. $\angle C \cong \angle A$	2. If two sides of a triangle are congruent, then the angles opposite these sides are congruent.

	3.	$\overline{BC} \cong \overline{AC}$	3. Hypothesis
	4.	$\angle A \cong \angle B$	4. Same as Reason number 2
	5.	$\angle C \cong \angle A \cong \angle B$	5. Transitive property for congruence
2.	1.	D is the midpoint of $\overline{BC}$.	1. Hypothesis
	2.	$BD = CD$	2. Definition of midpoint of a segment (1)
	3.	$\overline{BD} \cong \overline{CD}$	3. Definition of congruence of segments (2)
	4.	$\triangle ABC$ isosceles, with $\overline{AB} \cong \overline{AC}$	4. Hypothesis
	5.	$\overline{AD} \cong \overline{AD}$	5. Reflexive property of congruence
	6.	$\triangle ADB \cong \triangle ADC$	6. S.S.S. Postulate (3 , 4 , 5)
	7.	$\angle ADB \cong \angle ADC$	7. Corresponding parts of congruent triangles are congruent. (6)
	8.	$\angle ADB$ and $\angle ADC$ are right angles.	8. If the angles of a linear pair are congruent each is a right angle. (7)
	9.	$\overline{BC}$ is level.	9. By definition of level
3.	1.	$AC = CB = BA$	1. The measures of the sides of an equilateral triangle are equal.
	2.	$\frac{1}{2} AC = \frac{1}{2} CB = \frac{1}{2} BA$	2. Multiplication property of equality (1)
	3.	P , R , and Q are midpoints of sides $\overline{AC}$, $\overline{CB}$, $\overline{BA}$ respectively.	3. Hypothesis
	4.	$CP = PA = \frac{1}{2} AC$ $CR = RB = \frac{1}{2} CB$ $BQ = QA = \frac{1}{2} BA$	4. Definition of midpoint (3)

5. CP = PA = CR RB = BQ = QA or, arranging to better advantage,	5. Transitive property of equality (1 , 4)
6. $\overline{CP} \cong \overline{AQ} \cong \overline{BR}$ and $\overline{CR} \cong \overline{AP} \cong \overline{BQ}$	6. Segments with equal measures are congruent.(5)
7. $\angle C \cong \angle A \cong \angle B$	7. An equilateral triangle is equiangular. (1)
8. $\Delta CRP = \Delta APQ = \Delta BQR$	8. S.A.S. Postulate (6 , 7)
9. $\overline{RP} \cong \overline{PQ} \cong \overline{QR}$	9. Corresponding parts of congruent triangles are congruent. (8)
10. ΔPQR is equilateral	10. Definition of an equilateral triangle (9)

4.

1. $\overline{XY} \cong \overline{ZY}$ $\overline{XR} \cong \overline{ZR}$	1. Hypothesis
2. $\angle YXZ \cong \angle YZX$ $\angle RXZ \cong \angle RZX$	2. If two sides of a triangle are congruent, the angles opposite these sides are congruent. (1)
3. $m\angle YXZ = m\angle YZX$ $m\angle RXZ = m\angle RZX$	3. Definition of congruence of angles
4. $m\angle YXZ + m\angle RXZ = m\angle YZX + m\angle RZX$	4. Addition property of equality (3)
5. $m\angle YXZ + m\angle RXZ = m\angle YXR$ $m\angle YZX + m\angle RZX = m\angle YZR$	5. Betweenness-Angles Theorem
6. $m\angle YXR = m\angle YZR$	6. Substitution property of equality (4 , 5)
7. $\angle YXR \cong \angle YZR$	7. Angles with the same measure are congruent. (6)

5.

(a) Double-Column Form

1. $\overline{AC} \cong \overline{BC}$ or $AC = BC$	1. Hypothesis
2. $\frac{1}{2} AC = \frac{1}{2} BC$	2. Multiplication property of equality. (1)

3. $XC = \frac{1}{2} AC$ $YC = \frac{1}{2} BC$	3. Definition of midpoint
4. $XC = YC$	4. Substitution property of equality. (2 , 3)
5. $\angle CXY \cong \angle CYX$	5. If two sides of a triangle are congruent, the angles opposite those sides are congruent. (4)

(b) Paragraph Form.

Since X and Y are given the midpoints of $\overline{AC}$ and $\overline{BC}$ respectively, it follows that $XC = \frac{1}{2} AC$ and $YC = \frac{1}{2} BC$. But $AC = BC$ by hypothesis so, by the multiplicative property of equality,

$$\frac{1}{2} AC = \frac{1}{2} BC \quad \text{or} \quad XC = YC.$$

Then, in ΔXCY, $\angle CXY \cong \angle CYX$ since

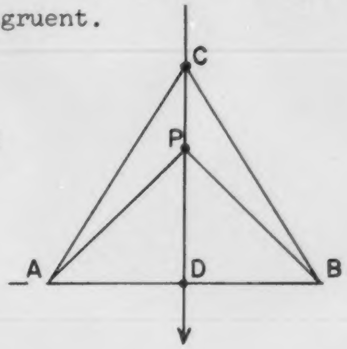
if two sides of a triangle are congruent, the angles opposite those sides are congruent.

6. Hypothesis:

In isosceles triangle ABC
$\overline{AC} \cong \overline{BC}$, $\overrightarrow{CD}$ bisects
$\angle ACB$, P is a point on
$\overrightarrow{CD}$.

Prove:

$PA = PB$



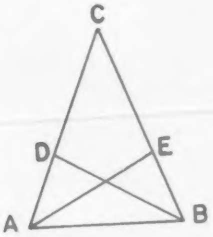
In ΔAPC and ΔBPC

1. $\overline{AC} \cong \overline{BC}$	1. Hypothesis
2. $\overline{CP} \cong \overline{CP}$	2. Reflexive property of congruence.
3. $\angle ACP \cong \angle BCP$	3. Hypothesis and definition of the bisector of an angle

4. $\Delta APC \cong \Delta BPC$	4. S.A.S. Postulate (1 , 2 , 3)
5. $PA = PB$	5. Definition of congruence for triangles (4)

7. Hypothesis:

ΔABC is isosceles with
$\overline{CA} \cong \overline{CB}$.
$\overline{AE}$ bisects $\angle CAB$
$\overline{BD}$ bisects $\angle CBA$



Prove: $\overline{AE} \cong \overline{BD}$

Prove: $\Delta ADB \cong \Delta BEA$

1. $\overline{AB} \cong \overline{BA}$	1. Reflexive property of congruence
2. $\overline{CA} \cong \overline{CB}$	2. Hypothesis
3. $\angle CAB \cong \angle CBA$ or $m\angle CAB = m\angle CBA$	3. If two sides of a triangle are congruent, the angles opposite those sides are congruent. (2)
4. $\frac{1}{2}\, m\angle CAB = \frac{1}{2}\, m\angle CBA$	4. Multiplication property of equality (3)
5. $m\angle EAB = \frac{1}{2}\, m\angle CAB$ and $m\angle DBA = \frac{1}{2}\, m\angle CBA$	5. Definition of angle bisector
6. $m\angle EAB = m\angle DBA$ or $\angle EAB \cong \angle DBA$	6. Substitution property of equality (4 , 5)
7. $\Delta ADB \cong \Delta BEA$	7. A.S.A. Postulate (1 , 2 , 6)
8. $\overline{AE} \cong \overline{BD}$	8. Corresponding parts of congruent triangles are congruent. (7)

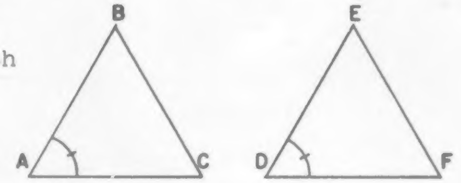
8. Hypothesis:

$\triangle ABC$ is isosceles with $\overline{AB} \cong \overline{CB}$.

$\triangle DEF$ is isosceles with $\overline{DE} \cong \overline{FE}$.

$AC = DF$

$m\angle A = m\angle D$



Prove: $\triangle BAC \cong \triangle EDF$

1. $\angle A \cong \angle C$ or $m\angle A = m\angle D$ $\angle D \cong \angle F$	1. Hypothesis and the Theorem: If two sides of a triangle are congruent the angles opposite those sides are congruent.
2. $\angle A \cong \angle D$	2. Hypothesis
3. $\angle C \cong \angle F$	3. Transitive property of congruence (1 , 2)
4. $\overline{AC} \cong \overline{DF}$ or $AC = DF$	4. Hypothesis
5. $\triangle BAC \cong \triangle EDF$	5. A.S.A. Postulate (2 , 3 , 4)

Problem Set 5-9

Again remind students that the validity of a statement must be supported by logical argument whereas a single counter example is sufficient to show that a statement is not valid.

In problem 4, consider the parts of the diagram as either coplanar or noncoplanar.

	Statement	Converse
(a)	If a number is positive, then it has one positive square root. Valid. In elementary algebra there is a theorem to the effect that every positive number has two square roots, one positive and one negative.	If a number has one positive square root, then it is a positive number. Valid. This follows from the fact that the square of a positive number is positive.
(b)	If a set of points is the interior of a triangle, then it is a convex set. Valid. The interior of a triangle is the intersection of three half-planes, and each half-plane is convex. The segment joining any two points in the interior of a triangle lies in each of the three half-planes, and hence, in their intersection.	If a set of points is a convex set, then it is the interior of a triangle. Not valid. One (of many) counter example is: The set of all points in the plane is a convex set which is not the interior of a triangle.
(c)	If two numbers are odd, their sum is odd. Not valid. Counter-example: 3 and 5 are each odd but their sum is even.	If the sum of two numbers is odd then each of the numbers is odd. Not valid. Counter-example: 5 and 4. Another counter example: $2 + \pi$ and $5 - \pi$.

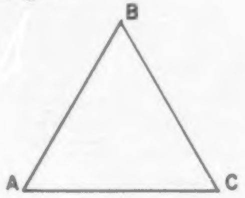
(d) If each angle in a set of angles is a right angle, then the angles in every pair of these angles are congruent. More simply: If two angles are right angles, then they are congruent.	If the angles in every pair of angles in a set of angles are congruent, then each of the angles is a right angle. Not valid. Counter-example: The angles in every pair of vertical angles are congruent, but they are not necessarily right angles.

2. Hypothesis:

In $\triangle ABC$,

$\angle A \cong \angle B \cong \angle C$

Prove: $\triangle ABC$ is equilateral.



1. $\angle A \cong \angle C$	1. Hypothesis
2. $\overline{BC} \cong \overline{BA}$	2. If two angles of a triangle are congruent, the sides opposite those angles are congruent. (1)
3. $\angle C \cong \angle B$	3. Same as 1.
4. $\overline{BA} \cong \overline{AC}$	4. Same as 2. (3)
5. $\overline{BC} \cong \overline{BA} \cong \overline{AC}$	5. Transitivity property of congruence (2 , 4)
6. $\triangle ABC$ is equilateral.	6. Definition of equilateral (5)

3.

1. $\angle m \cong \angle n$	1. Hypothesis
2. $\angle m$ and $\angle CBA$ form a linear pair. $\angle n$ and $\angle BCA$ are supplementary.	2. Hypothesis (since the diagram is part of the hypothesis) and the definition of a linear pair.

3.	$\angle m$ and $\angle CBA$ are supplementary. $\angle n$ and $\angle BCA$ are supplementary.	3.	If two angles form a linear pair, then they are supplementary. (2)
4.	$\angle CBA \cong \angle BCA$	4.	Supplements of congruent angles are congruent. (1 , 3)
5.	$\overline{AC} \cong \overline{AB}$	5.	If two angles of a $\triangle$ are congruent, the sides opposite those angles are congruent. (4)
6.	$\triangle ABC$ is isosceles.	6.	Definition of isosceles triangle (5)

4. Consider $\triangle HPF$ and $\triangle KPF$.

1.	$HF = KF$ and $HP = KP$	1.	Hypothesis
2.	$\overline{HF} \cong \overline{KF}$ and $\overline{HP} \cong \overline{KP}$	2.	Definition of congruence for segments (1)
3.	$\overline{FP} \cong \overline{FP}$	3.	Reflexive property of congruence
4.	$\triangle HPF \cong \triangle KPF$	4.	S.S.S. Postulate (2 , 3)
5.	$\angle x \cong \angle y$	5.	Corresponding parts of congruent triangles are congruent. (4)

5. Consider $\triangle ACE$ and $\triangle BCD$.

1.	$\angle y \cong \angle x$	1.	Hypothesis
2.	$\overline{CE} \cong \overline{DC}$	2.	If two angles of a triangle are congruent, the sides opposite those angles are congruent. (1)
3.	$AD = BE$ or $\overline{AD} \cong \overline{BE}$ and A , D , E , B are collinear.	3.	Hypothesis
4.	$\overline{AE} \cong \overline{BD}$	4.	Betweenness-Addition Theorem for Points (3)
5.	$\triangle ACE \cong \triangle BCD$	5.	S.A.S. Postulate (1 , 2 , 4)

6. $\overline{AC} \cong \overline{BC}$	6. Definition of congruence for triangles (5)
7. $\triangle ABC$ is isosceles.	7. Definition of isosceles triangle (6)

6.

Statements	Reasons
1. $\angle C \cong \angle F$	1. Hypothesis
2. $\overline{PQ} \perp \overline{AC}$ and $\overline{RS} \perp \overline{DF}$	2. Hypothesis
3. $\angle PQC$ is a right angle. $\angle RSF$ is a right angle.	3. Definition of perpendicular (2)
4. $\angle PQC \cong \angle RSF$	4. Right angles are congruent. (3)
5. $\overline{CQ} \cong \overline{FS}$	5. Hypothesis
6. $\triangle PQC \cong \triangle RSF$	6. A.S.A. Postulate (1 , 4 , 5)
7. $CP = FR$	7. Definition of congruence for triangles (6)

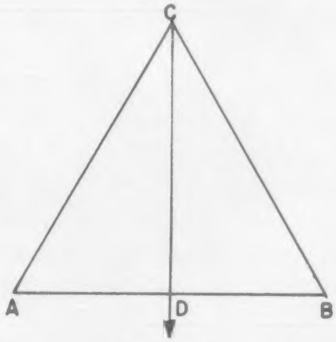
7. Hypothesis:

Isosceles $\triangle ABC$, with $\overline{CA} \cong \overline{CB}$, $\overrightarrow{CD}$ bisects $\angle C$ and intersects $\overline{AB}$ in point D.

Prove:

(a) $\overrightarrow{CD} \perp \overline{AB}$

(b) $AD = DB$



Statements	Reasons
1. $\angle ACD \cong \angle BCD$	1. Hypothesis and definition of angle bisector
2. $\overline{CA} \cong \overline{CB}$	2. Hypothesis
3. $\angle A \cong \angle B$	3. Base angles of an isosceles triangle are congruent. (2)
4. $\triangle ACD \cong \triangle BCD$	4. A.S.A. Postulate (1 , 2 , 3)

5. $\angle ADC \cong \angle BDC$	5. Corresponding parts of congruent triangles are congruent. (4)
6. $m\angle ADC = \angle BDC$	6. Congruent angles have equal measures. (5)
7. $\angle ADC$ and $\angle BDC$ form a linear pair.	7. Definition of linear pair
8. $\angle ADC$ and $\angle BDC$ are right angles.	8. Each of the angles of a linear pair which have the same measure is called a right angle. (6 , 7)
(a) 9. $\overrightarrow{CD} \perp \overline{AB}$	9. Definition of perpendicular (8)
(b)10. $AD = DB$	10. Definition of congruence for triangles (4)

Problem Set 5-10

The solutions for the problems in this set, except for Problem 7, are merely outlined or suggested.

1. Prove $\Delta ABC \cong \Delta ABD$ by S.A.S.
   Then $\overline{AC} \cong \overline{AD}$ and, in ΔACD, $\angle ACD \cong \angle ADC$
   since they are opposite the congruent sides.
2. $\Delta ABC = \Delta ABC'$ by A.S.A.
3. $\Delta DXA = \Delta EXA$ by S.A.S.
   Then $\overline{DX} \cong \overline{EX}$.
4. (a) A model is to be constructed.
   (b) Four faces, 12 face angles
   (c) All faces are congruent by S.S.S.
   All faces are equilateral triangles.
5. (a) Distances AB , AC and BC need not be equal since $\angle AVB$, $\angle BVC$ and $\angle CVA$ need not have the same measure.

   ΔAVB , ΔBVC and ΔCVA are each isosceles triangles but not necessarily congruent to each other.
   $\angle VAB \cong \angle VBA$, $\angle VBC \cong \angle VCB$, $\angle VCA \cong \angle VAC$

   (b) Then $AB = AC = BC$ and all six of the angles will be congruent.

6. Prove $\triangle NPB \cong \triangle NMC$ by S.A.S., first showing $PB = MC$ by the addition and substitution properties of equality and the Betweenness-Distance Theorem.
   Then $\overline{NB} \cong \overline{NC}$
   Prove: $\triangle ANB \cong \triangle ANC$ by S.A.S.
   Then $\overline{AB} \cong \overline{AC}$

7. From the definition of bisect for segments

$$\overline{AV} \cong \overline{EV} \quad ; \quad \overline{CV} \cong \overline{DV} \quad ; \quad \overline{BV} \cong \overline{FV}.$$

   From the definition of vertical angles and, since vertical angles are congruent,

$$\angle AVB \cong \angle EVF \quad ; \quad \angle CVA \cong \angle DVE \quad ; \quad \angle BVC \cong \angle FVD.$$

   Then by S.A.S.

$$\triangle AVB = \triangle EVF \quad ; \quad \triangle CVA = \triangle DVE \quad ; \quad \triangle BVC \quad ; \quad \triangle FVD.$$

   It follows that the corresponding parts are congruent and thus:

$$\overline{AB} \cong \overline{EF} \quad ; \quad \overline{CA} \cong \overline{DE} \quad ; \quad \overline{BC} \cong \overline{FD}.$$

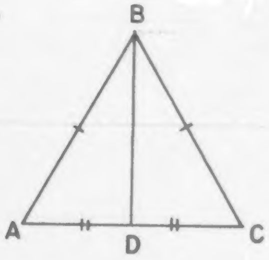
   Then $\triangle ABC \cong \triangle EFD$ by S.S.S.

8. Prove $\triangle ABD \cong \triangle ABC \cong \triangle DBC$ by S.A.S.
   It follows that $\overline{AD} \cong \overline{AC} \cong \overline{DC}$.
   Then, since $\triangle ADC$ is equilateral, it is also equiangular.

Problem Set 5-11

The following are merely the main ideas in the proof of the problems. The problems present no complication in proving congruence of triangles.

1.



(i) $\triangle ABC \cong \triangle CBD$ by S.S.S., thus $\angle ABD \cong \angle CBD$ and, by definition of midray, $\overrightarrow{BD}$ is a midray and $\overline{BD}$ bisects the vertex angle.

(ii) $\angle ADB \cong \angle CDB$, thus they are right angles, and then $\overline{BD} \perp \overline{AC}$.

2. If $\Delta ABC \cong \Delta DEF$, $AC \cong DF$, $\angle A \cong \angle D$, $AB = DE$, $AG = \frac{1}{2} AB$; $DH = \frac{1}{2} DE$.

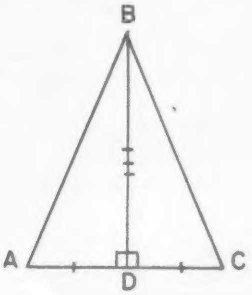
Then, $AG = DH$ by the multiplication and substitution properties of equality.
Hence, $\Delta ACG \cong \Delta DFH$ by S.A.S. and $\overline{CG} \cong \overline{FH}$.

3. $\angle AOB \cong \angle b$, $\angle COB \cong \angle a$ since vertical angles are congruent.

Then, since $\angle a \cong \angle b$, $\angle COB \cong \angle AOB$. $\overline{OB} \cong \overline{OB}$, and $\angle ABO \cong \angle CBO$ since they are right angles. It follows that $\Delta ABO \cong \Delta CBO$ by A.S.A. Thus $\overline{AO} \cong \overline{CO}$.

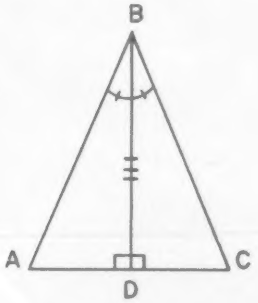
4. ΔABC is isosceles, so $\angle ABC \cong \angle ACB$.
$\angle EBC \cong \angle DCB$ since the measure of each is $\frac{1}{2}$ of the same measure. $\Delta CEB \cong \Delta BDC$ by A.S.A.
Thus $CE \cong BD$

5.



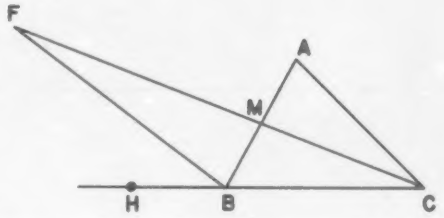
In ΔABD and ΔCBD $\overline{AD} \cong \overline{DC}$ because a median of a triangle bisects the side to which it is drawn. Since $\overline{BD} \perp \overline{AC}$, $\angle BDA$ and $\angle BDC$ are both right angles and are, therefore, congruent. $\overline{BD} \cong \overline{BD}$ by the reflexive property of congruence. Thus $\Delta ABD \cong \Delta CBD$ by S.A.S. Then $\overline{AB} \cong \overline{BC}$ and ΔABC is isosceles by the definition of an isosceles triangle.

6.



If $\overrightarrow{BD}$ is an angle bisector, $\angle ABD \cong \angle CBD$. $\overline{BD} \cong \overline{BD}$ by the reflexive property for congruence. Since $\overline{BD} \perp \overline{AC}$, $\angle BDA$ and $\angle BDC$ are right angles and thus are congruent. Therefore, $\Delta ABD \cong \Delta CBD$ by A.S.A. and $\overline{AB} \cong \overline{CB}$. Thus ΔABC is isosceles by the definition of isosceles triangle.

Problem Set 5-12a



1. Consider point H, any point on the ray opposite $\overrightarrow{BC}$. Let M be the midpoint of $\overline{BA}$ and let F be the point on the ray opposite $\overrightarrow{MC}$ such that $FM = CM$.

   Then $\Delta FMB \cong \Delta CMA$ by S.A.S. and $\angle FBM \cong \angle A$.

   Theorem 4-7 tells us that F is an interior point of $\angle HBA$.

   Therefore by the Betweenness-Angles Theorem $m\angle HBA = m\angle HBF + m\angle FBA$.

   Since $m\angle HBF > 0$, $m\angle HBA > m\angle FBA$ and therefore $m\angle HBA > m\angle A$.

   To prove $m\angle HBA > m\angle ACB$ start with the midpoint of $\overline{BC}$ rather than with the midpoint of $\overline{BA}$.

2. $m\angle ABC < 50$; $m\angle BAC < 50$; $m\angle ACB = 130$

3. By applying the Supplement Theorem (Th. 4-12) and the theorem on supplements of congruent angles (Th. 4-15) the congruence of the six exterior angles can be demonstrated. It is important that the student be able to identify and name these angles in his proof.

4. (a) $m\angle a < m\angle c$

   (b) $m\angle d < m\angle b$

   (c) $m\angle c < m\angle e$

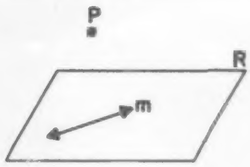
   (d) $m\angle a < m\angle c < m\angle e$

5. (a) The median must be perpendicular to the base. The median to the base of an isosceles triangle is perpendicular to the base by Theorem 5-11.

   (b) From Theorem 5-11 we know that given a line and a point not on the line, there is one and only one line which contains the given point and which is perpendicular to the given line.

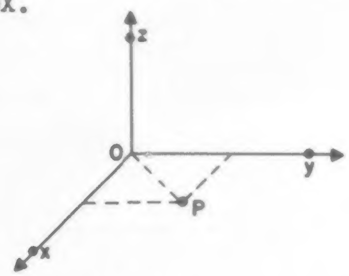
(c) Since $\overline{BD}$ is $\perp$ to $\overline{AC}$ by hypothesis and since the median from B is $\perp$ to $\overline{AC}$, and since there is only one line from $B \perp \overline{AC}$, $\overline{BD}$ and the median are the same line.

(d) If a segment from the vertex opposite the base of an isosceles triangle is perpendicular to the base, then that segment is the median of the triangle.

6. 

Yes. There is one and only one line which contains P and is perpendicular to m since if P is not in R, P is not in m and the relation is an example of Theorem 5-5.

7. There is one and only one line through P, perpendicular to $\overrightarrow{OZ}$.
There is one and only one line through P, perpendicular to $\overrightarrow{OY}$.
There is one and only one line through P, perpendicular to $\overrightarrow{OX}$.

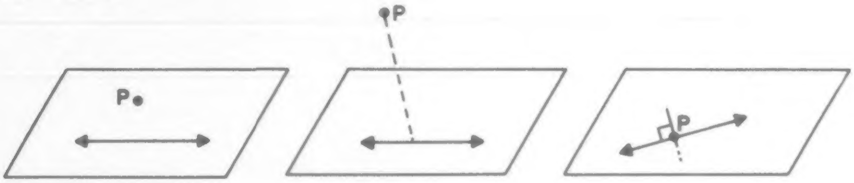


P lies in plane OXY.

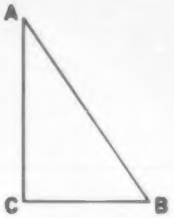
8. One and only one
One and only one

9. One and only one line contains P and is $\perp$ $\overleftrightarrow{RS}$.

10. Yes.



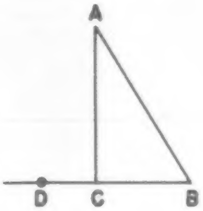
11.



Let $\angle C$ of $\triangle ABC$ be a right angle. We want to show that neither $\angle B$ nor $\angle A$ can be a right angle.

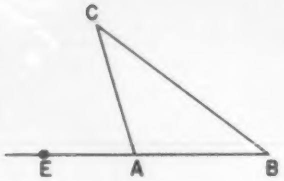
Assume $\angle B$ is a right angle. Then by definition of a perpendicular line $\overleftrightarrow{AC}$ and $\overleftrightarrow{AB}$ are both perpendicular to $\overleftrightarrow{CB}$. But this contradicts Theorem 5-11 which assures us there can be one and only one line containing A and $\perp$ to $\overleftrightarrow{CB}$. Therefore $\angle B$ cannot be a right angle. Likewise if $\angle A$ were a right angle $\overleftrightarrow{BA}$ and $\overleftrightarrow{BC}$ would both be $\perp$ to $\overleftrightarrow{AC}$ which again contradicts Theorem 5-11. Therefore $\angle A$ cannot be a right angle.

12.



Let $\angle ACB$ of $\triangle ACB$ be a right angle. We want to show that $\angle A$ and $\angle B$ are each acute angles.

Consider $\overrightarrow{CD}$ as opposite $\overrightarrow{CB}$. Then $\angle ACB$ and $\angle ACD$ form a linear pair and are supplementary and, since $m\angle ACB = 90$, $m\angle ACD$ also $= 90$. By Theorem 3-10, $m\angle ACD > m\angle B$ and $m\angle ACD > m\angle A$. Therefore $m\angle B < 90$ and $m\angle A < 90$ and hence, by the definition of an acute angle, $\angle B$ and $\angle A$ are both acute angles.

13. 

Let $\angle CAB$ of $\triangle CAB$ be an obtuse angle. We want to show that $\angle B$ and $\angle C$ are each acute angles.

Consider $\overrightarrow{AE}$ as opposite $\overrightarrow{AB}$. Then $\angle EAC$ and $\angle BAC$ form a linear pair and are, therefore, supplementary angles. By the definition of supplementary angles, $m\angle EAC + m\angle CAB = 180$ or $m\angle CAB = 180 - m\angle EAC$. But, by the hypothesis and the definition of an obtuse angle, $m\angle CAB > 90$. It follows that $180 - m\angle EAC > 90$ or, from the addition postulate of order, that $90 > m\angle EAC$.

But from Theorem 5-10, $m\angle EAC > m\angle B$ and $m\angle EAC > m\angle C$. Since $90 > m\angle EAC > m\angle C$ and $90 > m\angle EAC > m\angle B$ we get, by using the transitivity property of order, that $90 > m\angle C$ and $90 > m\angle B$. It follows from the definition of acute angle that $\angle B$ and $\angle C$ are both acute angles.

Problem Set 5-12b

You may wish to have students give only an outline of the proofs of these problems, as is given for most of the problems below.

1. The two triangles in which the diagonals are the corresponding parts are congruent by S.A.S.

2. No. The angles may be unequal even though the sides are equal and the polygon would not then satisfy the definition of a regular polygon.

   Example: a rhombus,

3. $\triangle BAD \cong \triangle DAC \cong \triangle CAB$ from the hypothesis and S.A.S.
   $BD = DC = CB$ by definition of congruence for triangles.
   $\triangle BDC$ is equilateral by definition of equilateral triangle.
   $\triangle BDC$ is equiangular since every equilateral triangle is equiangular.

4. Since in the figure segments with a common endpoint are collinear, the figure does not satisfy condition (2) in the definition of a polygon. To be a member of a subset of polygons, a figure must first be a member of the set of polygons.

5. Outline of proof.

   $\triangle FAB \cong \triangle BCD \cong \triangle DEF$ by definition of regular polygon and S.A.S.

   $FB = BD = DF$ by definition of congruence for triangles.

   $\triangle FBD$ is equilateral by definition of equilateral triangle.

6. Outline of proof.

   1. $\triangle FBD$ is an equilateral triangle as proved in Problem 5 above.
   2. $\triangle FAD \cong \triangle BAD$ from hypothesis, Step 1 and S.S.S.
   3. $\angle FDA \cong \angle BDA$ by definition of congruence for triangles.
   4. $\overline{AD} \perp \overline{FB}$ because a line which bisects the vertex angle of an isosceles triangle is perpendicular to the base.

7. Outline of Proof.

   1. $\triangle BAD \cong \triangle CAD$ from hypothesis and S.S.S.
   2. $\angle BAD \cong \angle CAD$ by definition of congruence for triangles.
   3. $\triangle EAG \cong \triangle FAG$ from hypothesis, multiplication property of order, Step 2, S.A.S.
   4. $EG = FG$ by definition of congruence for triangles.
   5. $\triangle EFG$ is an isosceles triangle by definition of isosceles triangle.

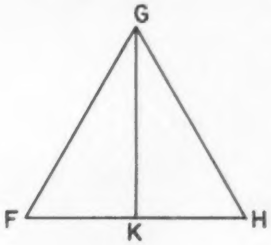
Chapter 5

Review Problems

Sketches or brief outlines of the proofs are given.

1. Prove $\triangle GCE \cong \triangle FBE$ by A.S.A.
It follows that $\overline{CE} \cong \overline{BE}$ and thus, by definition of bisects for segments, $\overline{GF}$ bisects $\overline{CB}$.

2. $\overline{AB} \cong \overline{CA}$, $\overline{AC} \cong \overline{CB}$, $\overline{BC} \cong \overline{AB}$ by definition of equilateral triangle. Then $\triangle ABC \cong \triangle CAB$ by S.S.S.

3. 

$\triangle FGK \cong \triangle HGK$ by A.S.A.
$\overline{FG} \cong \overline{HG}$ by definition of congruence for triangles.
Therefore $\triangle FGH$ is isosceles by definition of isosceles triangle.

4. $\triangle APQ \cong \triangle BPQ$ by S.S.S.
Then $\angle APQ \cong \angle BPQ$ from the definition of congruence for triangles.
The proof holds if A is not in the plane of P, Q and B.

5. $m\angle HAB = m\angle HBA$ since if two sides of a triangle are congruent, the angles opposite those sides are congruent or have equal measures. $m\angle FAB = \frac{1}{2} m\angle HAB = \frac{1}{2} m\angle HBA = m\angle FBA$ from the definition of bisect for angles, and the multiplication property of equality.
Then $m\angle FAB = m\angle FBA$ from the transitive property of equality.
Then $AF = BF$ since if two angles of a triangle have the same measure, the sides opposite those angles have the same measure.

6. Prove $\triangle AED \cong \triangle BCD$ by S.A.S.
It follows that $\overline{AD} \cong \overline{BD}$ and thus that $\angle DAB \cong \angle DBA$.

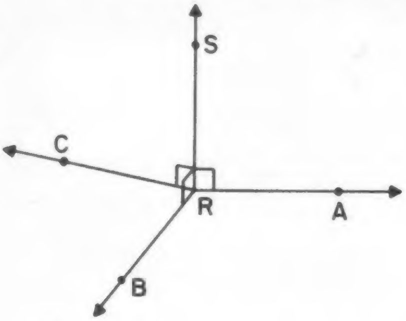
7. $\triangle ARF \cong \triangle BFR$ S.S.S. Therefore $\angle ARF \cong \angle BFR$. The figure need not lie in a plane.

8. (a) $\angle AXF \cong \angle BXR$ by the Betweenness-Addition Theorem for Rays.

$\triangle AXF \cong \triangle BXR$ by A.S.A. Therefore $\overline{AF} \cong \overline{BR}$.

(b) Yes. Since the betweenness concept for rays is needed, the figure must lie in a plane.

9.



$\triangle SRA \cong \triangle SRB \cong \triangle SRC$ by S.A.S. Thus $SA = SB = SC$.

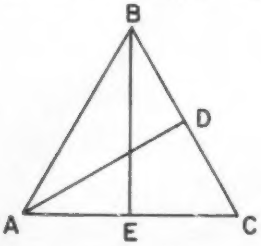
10. $TP = QX$;
Use the Vertical Angle Theorem and then the A.S.A. Postulate.

11. $\angle QBF \cong \angle WHF$, Complements of congruent angles are congruent.
$\triangle BFQ \cong \triangle HFW$ by S.A.S.

12. $\angle BAH = \angle BAF$ and $\angle RAH = \angle RAF$ since F, H, A are collinear.
Then $\angle BAF \cong \angle RAF$ and $\triangle BAF \cong \triangle RAF$ by S.A.S.
It follows that $FB = FR$.

13. $\triangle SRQ \cong \triangle SXQ$ by S.A.S. Therefore $\angle R \cong \angle SXQ$.

14.



$\triangle ABE \cong \triangle CBE$ by S.A.S. Therefore $\overline{AB} \cong \overline{CB}$.

$\triangle CAD \cong \triangle BAD$ by S.A.S. Therefore $\overline{AC} \cong \overline{AB}$.

$\overline{AC} \cong \overline{AB} \cong \overline{CB}$ by the transitivity property of equality.

It follows that $\triangle ABC$ is an equilateral triangle from the definition of equilateral triangle.

15. $\overline{BF} \cong \overline{RQ}$ by the multiplication property of equality.
$\triangle ABF \cong \triangle HRQ$ by S.S.S.
Therefore $\angle B \cong \angle R$ and thus $\triangle ABC \cong \triangle HRW$ by S.A.S.

16. $\angle RBF \cong \angle RFB$ since in $\triangle BRF$ the opposite sides are congruent. $BH = FA$ by the Betweenness-Addition Theorem for Points since A, B, F, H, are collinear.
$\triangle BRH \cong \triangle FRA$ by S.A.S., therefore $\angle BRH \cong \angle FRA$.
A student might prove $\triangle ABR \cong \triangle HFR$ by S.A.S. after proving $\angle ABR \cong \angle HFR$ by noting that they are supplements of congruent angles. Then $\angle ARB \cong \angle HRF$ and since $\overrightarrow{RB}$ and $\overrightarrow{RF}$ are between $\overrightarrow{RA}$ and $\overrightarrow{RH}$, it would follow that $\angle BRH \cong \angle FRA$ by the Betweenness-Addition Theorem for Rays.

17. $\triangle FWA \cong \triangle HBA$ by A.S.A., therefore $FW = HB$.

18. $\angle GKR \cong \angle HRK$ since they are supplements of congruent. angles, $\angle a$ and $\angle b$. $\overline{KR} \cong \overline{RK}$, therefore $\triangle GKR \cong \triangle HRK$ by A.S.A. Thus $\overline{GR} \cong \overline{HK}$.

19. $AF = HB$ since each is $\frac{2}{3}$ AH.
$\triangle RFA \cong \triangle QBH$ by S.A.S.
$\angle RFA \cong \angle QBH$, thus $BW = FW$ by the converse of the Isosceles Triangle Theorem.

20. (a) $\triangle AQP \cong \triangle BQP$ by S.S.S.
Then $\angle AQR \cong \angle BQR$
and $\triangle AQR = \triangle BQR$ by S.A.S.
Thus $RA = RB$.

(b) The five points need not be coplanar. The proof will hold whether or not A is in the same plane as points B, R, P, Q.

21. $\angle HRA \cong \angle BRA$ since they are supplements of congruent angles.
$\triangle HRA \cong \triangle BRA$ by A.S.A.
$\overline{HA} \cong \overline{BA}$ and $\triangle HAB$ is isosceles.
Then $\overline{AF}$, the bisector of the vertex angle, is $\perp$ to the base $\overline{HB}$.

22. $\triangle RBA \cong \triangle FBH$ by A.S.A. Thus $\angle A \cong \angle H$.
To prove $\overline{AM} \cong \overline{HM}$:
(1) Consider $\triangle BAH$. $\angle BAH \cong \angle BHA$ by the Isosceles Triangle Theorem.
Then, since $\overrightarrow{AF}$ is between $\overrightarrow{AR}$ and $\overrightarrow{AH}$, and $\overrightarrow{HB}$ is between $\overrightarrow{HA}$ and $\overrightarrow{HF}$, $\angle RAH \cong \angle FHA$ by the Betweenness-Angles Theorem and properties of equality. It follows that $\overline{AM} \cong \overline{HM}$.
(2) Otherwise, prove $\angle MRH \cong \angle MFA$ by supplements of $\cong \angle s$; $RH = FA$ by the addition property of equality.
Then $\triangle AMF \cong \triangle HMR$ by A.S.A. and $\overline{AM} \cong \overline{HM}$.

23. $\angle BQF \cong \angle HQF$ from the hypothesis and the multiplication property of equality.
$\triangle BQF \cong \triangle HQF$ by A.S.A.
Thus $\overline{BQ} \cong \overline{HQ}$.

24. $XR = QW$ by the Betweenness-Addition Theorem for Points.
$\triangle XAR \cong \triangle QMW$ by A.S.A.
Thus $XA = QM$ and,
since $\angle X \cong \angle Q$, $XK = QK$.
Then by the addition property of equality, $KA = KM$.

25. $\triangle BRQ \cong \triangle TRS$ by A.S.A.
Thus $BR = TR$ and, since $\angle BRX \cong \angle TRY$ (vertical $\angle s$),
$\triangle BRX \cong \triangle TRY$ and thus $RX = RY$.
(Note: $\triangle RXQ$ and $\triangle RYS$ could have been used.)

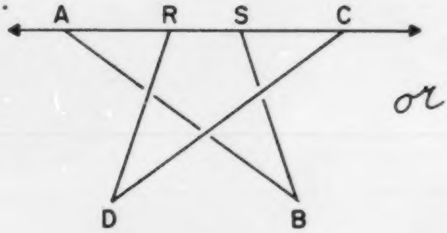
26. $\triangle ARQ \cong \triangle ASQ$ by S.A.S.
Thus $\overline{AR} \cong \overline{AS}$.
$\triangle RCA \cong \triangle SCA$ by S.S.S. Thus $\angle RCA \cong \angle SCA$.

27. No. We have two triangles in which two sides and an angle opposite one of them in one triangle are congruent to the corresponding parts of the other triangle. Our triangle postulates, S.S.S., A.S.A., S.A.S., do not apply.
Two sides and an angle opposite one of them do not necessarily give a unique triangle.

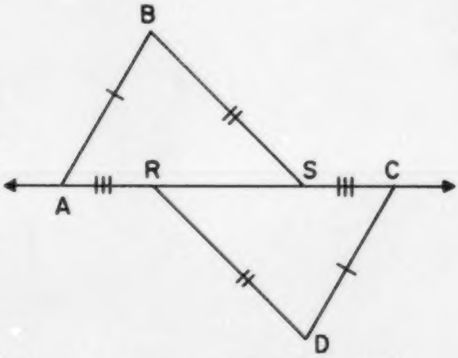
28. $\angle TJX \cong \angle BTX$ by the Betweenness-Angles Theorem and properties of equality.
$\triangle TJX \cong \triangle BJX$ by S.A.S.
Then $\angle TXJ \cong \angle BXJ$ and
$\triangle PJX \cong \triangle QJX$ by A.S.A.
Therefore $\angle x \cong \angle y$.

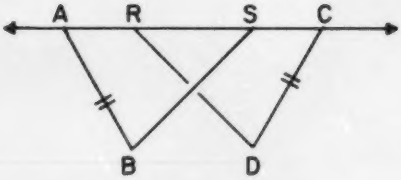
(Note: Student may prove $\angle T \cong \angle B$ to get $\triangle TJP \cong \triangle BJQ$. Then $\angle JPT \cong \angle JQB$ and $\angle x$ would be $\cong$ $\angle y$ since they are supplements of congruent angles.)

29.



or





(a) AS = RC by the Betweenness-Addition Theorem for Points. $\triangle BSA \cong \triangle DRC$ by S.S.S. Thus $\angle BSA \cong \angle DRC$.

(b) No. B need not be in the same plane as $\overleftrightarrow{AC}$ and D.

30. (1) Prove $\triangle ABM \cong \triangle FBR$ by S.A.S. Then $\angle A \cong \angle F$.

(2) Prove $\triangle AQR \cong \triangle FQM$ by A.S.A.

a. Use the Betweenness-Distance Theorem to show $AR = MF$.

b. Use supplements of congruent angles are congruent to show $\angle ARQ \cong \angle FMQ$.

Refer to Problem 22 for a suggestion which will help prove $\triangle AQR \cong \triangle FQM$ by S.A.S.

31. Show $\triangle ADC \cong \triangle GDF$ by S.A.S.
Then $\overline{AC} \cong \overline{GF}$.
Show $\triangle CDB \cong \triangle FDE$ by S.A.S.
Then $\overline{CB} \cong \overline{FE}$.
Show $\triangle ADB \cong \triangle GDE$ by S.A.S.
Then $\overline{AB} \cong \overline{GE}$.
Show $\triangle BCA \cong \triangle EFG$ by S.S.S.

32. Yes. No part of the proof depends upon the segments being coplanar.
For a three dimensional picture of the same problem refer to Problem 7 of Problem Set 5-10.

33. (1) Prove $\triangle AFB \cong \triangle MFH$ by S.A.S.
Then $\angle A \cong \angle M$ and $\overline{AB} \cong \overline{MH}$.

(2) Prove $\triangle ABT \cong \triangle MHR$ by S.A.S.

a. Use the multiplication property of equality to show $AT = MR$.

34. (1) Prove $\triangle QAX \cong \triangle PAX$ by S.A.S.
Then $\overline{QX} \cong \overline{PX}$ and $\triangle PQX$ is isosceles.

35. (1) Prove $\triangle ACF \cong \triangle ABH$ by S.A.S.
Then $\overline{CF} \cong \overline{BH}$.

(2) Prove $\triangle BCF \cong \triangle CBH$ by S.S.S.
Then $\angle FBC \cong \angle HCB$.

(3) Prove $\angle ABC \cong \angle ACB$ by showing them supplements of congruent angles.

REVIEW PROBLEMS

Chapters 1-5

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Chapter 6

ANSWERS AND SOLUTIONS

Problem Set 6-2

The first nine problems in this set are designed for oral class discussion to help reinforce the meaning of the definitions in this section. Problem 16 should be done by all students in the class. However, a teacher may wish to assign Problem 17 to half the students in the class and Problem 18 to the other half of the students. Students in these two groups should share their findings.

Problem Set 6-2

1. (a) Alternate interior.
   (b) Consecutive interior.
   (c) QPE (or, EPQ) .
   (d) PQB (or, BQP) .

2. (a) Vertical.
   (b) Alternate interior.
   (c) Corresponding.
   (d) Corresponding.
   (e) c .
   (f) u .
   (g) Vertical.
   (h) Adjacent supplementary (or, supplementary).
   (i) Two.
   (j) Two.
   (k) Four.

3. (a) $\angle FDE$ and $\angle DBC$.
   (b) $\angle ABD$ and $\angle EDB$.
   (c) $\angle EDB$ and $\angle CBD$.

4. (a) $\angle XYR$ and $\angle R$
   $\angle ZYW$ and $\angle W$.
   (b) $\angle R$ and $\angle RYZ$
   $\angle W$ and $\angle XYW$
   $\angle R$ and $\angle W$
   $\angle RYW$ and $\angle R$
   $\angle RYW$ and $\angle W$
   (c) There are none.

5. (a) Corresponding angles.
(b) Consecutive interior angles.
(c) Alternate interior angles.
(d) Corresponding angles.

6. (a) $\angle CDB$ and $\angle EBD$,
$\angle CDB$ and $\angle DBA$.
(b) $\angle BCD$ and $\angle CDB$,
$\angle DCB$ and $\angle CBD$,
$\angle CBD$ and $\angle BDC$,
$\angle DCB$ and $\angle CBE$.
(c) $\angle BCD$ and $\angle ABD$,
$\angle BCD$ and $\angle ABE$.

7. (a) $\angle a$, $\angle e$ $\quad$ $\angle i$, $\angle n$
$\angle b$, $\angle f$ $\quad$ $\angle j$, $\angle p$
$\angle c$, $\angle g$ $\quad$ $\angle k$, $\angle q$
$\angle d$, $\angle h$ $\quad$ $\angle m$, $\angle r$
$\angle a$, $\angle i$ $\quad$ $\angle e$, $\angle n$
$\angle b$, $\angle j$ $\quad$ $\angle f$, $\angle p$
$\angle c$, $\angle k$ $\quad$ $\angle g$, $\angle q$
$\angle d$, $\angle m$ $\quad$ $\angle h$, $\angle r$
(b) $\angle c$, $\angle f$ $\quad$ $\angle k$, $\angle p$
$\angle d$, $\angle e$ $\quad$ $\angle m$, $\angle n$
$\angle h$, $\angle n$ $\quad$ $\angle d$, $\angle i$
$\angle f$, $\angle q$ $\quad$ $\angle b$, $\angle k$
(c) $\angle d$, $\angle f$ $\quad$ $\angle d$, $\angle k$
$\angle k$, $\angle n$ $\quad$ $\angle f$, $\angle n$
$\angle b$, $\angle i$ $\quad$ $\angle c$, $\angle e$
$\angle h$, $\angle q$ $\quad$ $\angle m$, $\angle p$

8. (a) Corresponding angles:

$\angle WXP$ and $\angle QPZ$,
$\angle YXP$ and $\angle RPZ$.

Alternate interior angles:

$\angle WXP$ and $\angle RPX$,
$\angle QPX$ and $\angle YXP$,
$\angle WXZ$ and $\angle YZX$,
$\angle QPZ$ and $\angle YZP$.

Consecutive interior angles:

$\angle WXP$ and $\angle QPX$,
$\angle YXP$ and $\angle RPX$,
$\angle RPZ$ and $\angle RZP$,
$\angle YXZ$ and $\angle YZX$.

(b) Corresponding angles:

$\angle YRP$ and $\angle XPQ$,
$\angle ZRP$ and $\angle ZPQ$.

Alternate interior angles:

$\angle ZRP$ and $\angle XPR$,
$\angle YRP$ and $\angle ZPR$.

Consecutive interior angles:

$\angle ZPR$ and $\angle ZRP$,
$\angle XPR$ and $\angle YRP$.

(c) Corresponding angles:

$\angle ZYX$ and $\angle ZXW$.

Alternate interior angles:
None.

Consecutive interior angles:

$\angle ZXY$ and $\angle ZYX$.

9.

	Lines	Transversal
(a)	$\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.	$\overleftrightarrow{BD}$.
(b)	$\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$.	$\overleftrightarrow{BD}$.
(c)	$\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.	$\overleftrightarrow{AD}$.
(d)	$\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$.	$\overleftrightarrow{CD}$.

*10. (a) 50 .
(b) 130 .

*11. If $\overleftrightarrow{XZ} \perp \overleftrightarrow{CD}$, then $m \angle a = 90 = m \angle b$ and $\overleftrightarrow{RW} \perp \overleftrightarrow{CD}$.

*12. If $\overleftrightarrow{AB} \perp \overleftrightarrow{EF}$ and $\overleftrightarrow{CD} \perp \overleftrightarrow{EF}$, then

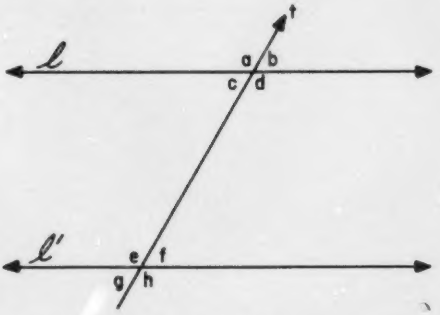
(a) $m \angle x = 90 = m \angle y$.
(b) $m \angle x = 90 = m \angle z$.

*13. If $m \angle a = 100 = m \angle b$, then $m \angle c = 80 = m \angle d$.

*14. If $m\angle a = 70$, then $m\angle c = 70$.
If $m\angle a = 70 = m\angle b$, then $m\angle d = 70$.

*15. If $m\angle a = 110$, then $m\angle b = 70$, $m\angle c = 110$, $m\angle d = 70$. If $m\angle d = 70 = m\angle e$, then $m\angle g = 70$, $m\angle f = 110$, and $m\angle h = 110$.

*16. Hypothesis:
ℓ and ℓ'
and transversal t.
$\angle c \cong \angle f$.



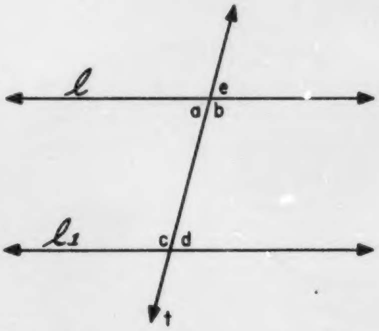
(a) $\angle d$ is a supplement of $\angle c$ and $\angle e$ is a supplement of $\angle f$ by the Supplement Theorem (Theorem 4-13). Since $\angle c \cong \angle f$, then $\angle d \cong \angle e$ because supplements of congruent angles are congruent.

(b) $\angle e$ is a supplement of $\angle f$ and $\angle f \cong \angle c$; therefore $\angle e$ is a supplement of $\angle c$. Similarly, $\angle f$ is a supplement of $\angle d$.

(c) $\angle b \cong \angle c$, vertical angles are congruent. Since $\angle c \cong \angle f$, then it follows that $\angle b \cong \angle f$. Similarly, $\angle a \cong \angle e$, $\angle c \cong \angle g$, $\angle d \cong \angle h$.

*17. Hypothesis:

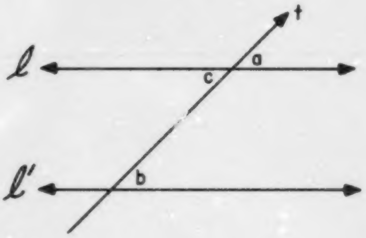
Transversal t to
ℓ, ℓ_1
$\angle a$ is supplementary to $\angle c$.



(a) By hypothesis, $\angle a$ is supplementary to $\angle c$; $\angle b$ and $\angle a$ are supplementary since they are a linear pair. Therefore, we have $\angle b \cong \angle c$.

(b) and (c) follow from (a) as a result of Problem 16.

*18. (a) By hypothesis ℓ and ℓ' and transversal t are such that $\angle a \cong \angle b$. This can be done by using congruent vertical angles to prove that the alternate interior angles are congruent.



(b) and (c) follow from (a) as a result of Problem 16.

Problem Set 6-3

1. (a) Converse of Theorem 6-1. Let two distinct coplanar lines be given. If the lines are parallel, then there is a transversal of the lines which is perpendicular to each of them.

Contrapositive of Theorem 6-1. Let two distinct coplanar lines be given. If the lines are not parallel, then a transversal of the lines cannot be perpendicular to both of them.

(b) The Contrapositive can be accepted as true because it is logically equivalent to the theorem.

2. (a) Converse. If two angles of a triangle are congruent, the sides opposite these angles are congruent.

Contrapositive. If two angles of a triangle are not congruent, then the sides opposite these angles are not congruent.

(b) Both can be accepted as true. The converse was proved in Chapter 5; and the contrapositive is logically equivalent to the statement, which was also proved in Chapter 5.

*3. If two distinct coplanar lines and a transversal of them determine a pair of alternate interior angles which are not congruent, then the two lines are not parallel.

Problem Set 6-4

This is a lengthy problem set, and teachers should consider the problems very carefully before making an assignment. Problems 4 through 7 and Problems 19 and 20 might be conveniently used as oral exercises. The problems from 8 through 18 increase in difficulty. Only the better students in the class should be expected to do Problems 15 through 18.

1. Corollary 6-2-1.

   Proof: Two coplanar lines are given. By hypothesis, a pair of corresponding angles determined by a transversal are congruent. Hence [see Remark below] a pair of alternate interior angles determined by the same transversal are congruent. By Theorem 6-2, the given lines are parallel.

   [Remark: The proof of this corollary illustrates the importance of Problem 18 in Problem Set 6-2. If a class needs further stress on details, then the word "Hence" above may be replaced by a proof of Part (a) of Problem 18.]

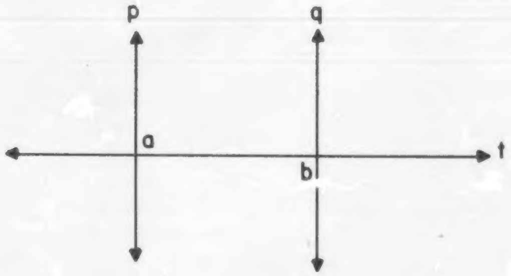
2. Corollary 6-2-2.

   Proof: Two coplanar lines are given. By hypothesis, a pair of consecutive interior angles determined by a transversal are supplementary. Hence a pair of alternate interior angles determined by the same transversal are congruent. By Theorem 6-2, the given lines are parallel.

   [Remark: Corollary 6-2-2 utilizes Problem 17 in Problem Set 6-2 in the same manner as Corollary 6-2-1 exploits Problem 18. Compare the Remark in Problem 1 of this set.]

3. Theorem 6-1.

Proof: Given coplanar lines p , q , and t , such that $p \perp t$ and $q \perp t$. We are required to prove $p \parallel q$.



Statements	Reasons
1. $p \perp t$, $q \perp t$.	1. Hypothesis
2. $\angle a$ is a right angle, $\angle b$ is a right angle.	2. If the sides of an angle are perpendicular, the angle is a right angle.
3. $\angle a \cong \angle b$.	3. Right angles are congruent.
4. $p \parallel q$.	4. Theorem 6-2.

4. (a) Corollary 6-2-1.
(b) Theorem 6-2.
(c) Corollary 6-2-1.
(d) Corollary 6-2-2.

5. (a) $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$ Theorem 6-2.
(b) $\overleftrightarrow{CD} \parallel \overleftrightarrow{AB}$ Theorem 6-2.
(c) $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$ Corollary 6-2-2.
(d) $\overleftrightarrow{CD} \parallel \overleftrightarrow{AB}$ Corollary 6-2-2.

6. $\overleftrightarrow{AB} \parallel \overleftrightarrow{DE}$ Theorem 6-1.

7. $x \parallel y$ Theorem 6-1.
$a \parallel c$ Theorem 6-2.

8. $4x - 12 + 5x + 30 = 180$
$x = 18$
$4x - 12 = 60$
$5x + 30 = 120$
$3x + 6 = 60$
$m \parallel n$ Corollary 6-2-1 or Corollary 6-2-2.

9. $3x + 2x + 20 = 180$
$x = 32$
$2x + 20 = 84$
$3x = 96$
$4x - 10 = 118$
m is not parallel to n. A pair of alternate interior angles are not congruent.

10. $\angle B \cong \angle C$.
$\angle C \cong \angle ADE$.
$\angle B \cong \angle ADE$.
$\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$ Corollary 6-2-1.

11. $\triangle ACD \cong \triangle CAB$ by S.S.S.
$\angle DAC \cong \angle BCA$, thus $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$.
$\angle BAC \cong \angle DCA$, thus $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$.

12. $\triangle AOB \cong \triangle COD$ by S.A.S., and $\angle BAO \cong \angle DCO$, thus $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ by Theorem 6-2.

$\triangle AOD \cong \triangle COB$ by S.A.S., and $\angle DAO \cong \angle BCO$, thus $\overleftrightarrow{AD} \parallel \overleftrightarrow{CB}$ by Theorem 6-2.

13. $\angle YXA$ and $\angle ZYC$ are right angles. $\angle AXS \cong \angle CYT$ by hypothesis. Therefore $\angle YXS \cong \angle ZYT$, since complements of congruent angles are congruent. $\overleftrightarrow{XS} \parallel \overleftrightarrow{YT}$ by Corollary 6-2-1.

14. (a) $\triangle XAZ$ is isosceles and $\angle X \cong \angle AZX$.
$\triangle WBY$ is isosceles and $\angle W \cong \angle BYW$.
$\angle AZX \cong \angle X \cong \angle W \cong \angle BYW$. Therefore $\overleftrightarrow{AZ} \parallel \overleftrightarrow{BY}$ by Theorem 6-2.
(b) No. $\overleftrightarrow{AZ}$ and $\overleftrightarrow{BY}$ must lie in the same plane to be parallel.

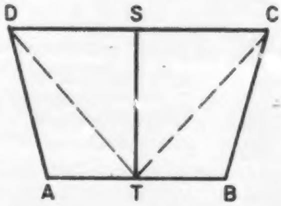
15. $\triangle ABD \cong \triangle BAC$ by S.A.S. and $DB = CA$. Then $\triangle DCB \cong \triangle CDA$ by S.S.S. and $\angle BCD \cong \angle ADC$.

It is not possible to prove that $\angle BCD$ and $\angle ADC$ are right angles. (Attempts to do this suggest a need for some further postulate.)

16. Proof: $\triangle APR \cong \triangle PBQ \cong \triangle RQC \cong \triangle QRP$ by S.S.S. By corresponding parts $m\angle a = m\angle A$, $m\angle b = m\angle B$ and $m\angle c = m\angle C$. Since the sum of the measures of $\angle a$, $\angle b$ and $\angle c$ is 180 by Theorem 4-9, the sum of the measures of $\angle A$, $\angle B$, and $\angle C$ is 180 .

17. Proof: $\triangle PAR \cong \triangle QAR$ by S.A.S. Then $\angle ARP \cong \angle ARQ$ and $\overleftrightarrow{AR} \perp \overleftrightarrow{PQ}$. By a similar proof using $\triangle ABD$ and $\triangle ACD$, $\overleftrightarrow{AD} \perp \overleftrightarrow{BC}$. Then $\overleftrightarrow{PQ} \parallel \overleftrightarrow{BC}$ by Theorem 6-1.

18.



Statements	Reasons
1. $\triangle DAT \cong \triangle CBT$.	1. S.A.S.
2. $DT = CT$.	2. Definition of congruence of triangles.
3. $m\angle DTA = m\angle CTB$.	3. Definition of congruence of triangles.
4. $m\angle DTS = m\angle CTS$.	4. Theorem 5-8.
5. $m\angle STA = m\angle STB$.	5. Betweenness-Addition Theorem.
6. $\overleftrightarrow{ST} \perp \overleftrightarrow{AB}$.	6. Theorem 4-11.
7. $\overleftrightarrow{ST} \perp \overleftrightarrow{CD}$.	7. Theorem 5-8.
8. $\overleftrightarrow{DC} \parallel \overleftrightarrow{AB}$.	8. Theorem 6-1.

An alternate plan, which does not take advantage of Theorem 5-8, is the following. From Step 1, deduce that $\angle DTA \cong \angle CTB$ and that $\overline{DT} \cong \overline{CT}$. Prove that $\triangle DST \cong \triangle CST$ by S.S.S. Deduce that $\angle DTS \cong \angle CTS$. Then $\angle ATS \cong \angle BTS$, and Theorem 4-11 establishes Step 6. The fact that $\triangle DST \cong \triangle CST$ yields $\angle DST \cong \angle CST$, and Step 7 becomes a second application of Theorem 4-11. Step 8, as before, completes the proof.

*19. Contrapositive of Corollary 6-2-1: If two coplanar lines are not parallel, then any two corresponding angles determined by a transversal of the lines are not congruent.

Contrapositive of Corollary 6-2-2: If two coplanar lines are not parallel, then any two consecutive interior angles determined by a transversal of the lines are not supplementary.

These statements can be accepted as true at this time because the contrapositive of a statement is logically equivalent to the statement.

*20. *Converse of Theorem 6-2*. If two distinct lines are parallel, then any two alternate interior angles determined by a transversal of the lines are congruent.

Converse of Corollary 6-2-1. If two distinct lines are parallel, then any two corresponding angles determined by a transversal of the lines are congruent.

Converse of Corollary 6-2-2. If two distinct lines are parallel, then any two consecutive interior angles determined by a transversal of the lines are supplementary.

No, the converse of a theorem needs to be proved before it can be accepted as true.

Problem Set 6-6a

1. Corollary 6-4-1.

   Proof: By hypothesis, two parallel lines and a transversal of them are given. By Theorem 6-4, any two alternate interior angles are congruent. Hence [see Remark below] any two corresponding angles are congruent.

   [Remark: Compare the Remark on the solution of Problem 1 in Problem Set 6-4. The proof of Corollary 6-4-1 illustrates the importance of Problem 16 (c) in Problem Set 6-2.]

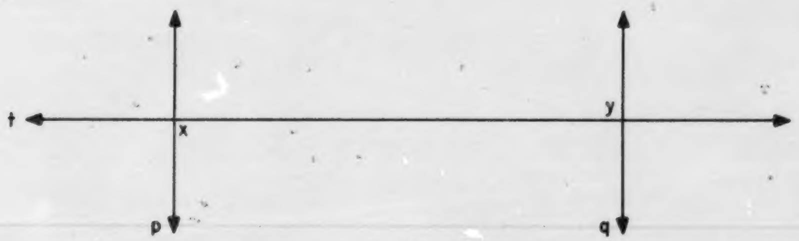
2. Corollary 6-4-2.

   Proof: By hypothesis, two parallel lines and a transversal of them are given. By Theorem 6-4, any two alternate interior angles are congruent. Hence any two consecutive interior angles are supplementary.

   [Remark: Compare Problem 1 above and Problem 16 (b) in Problem Set 6-2.]

3. Corollary 6-4-3.

   Proof: Given $p \parallel q$ and $p \perp t$. We are required to prove $q \perp t$.



	Statements		Reasons
1.	$p \parallel q$.	1.	Hypothesis.
2.	$\angle y \cong \angle x$.	2.	Theorem 6-4.
3.	$p \perp t$.	3.	Hypothesis.
4.	$m \angle x = 90$.	4.	If two lines are perpendicular, they determine a right angle.
5.	$m \angle y = 90$.	5.	Transitive property of equality.
6.	$q \perp t$.	6.	Definition of perpendicular lines.

4. (a) $m \angle b = 100$. Theorem 6-4.
   (b) $m \angle c = 80$. Corollary 6-4-2.
   (c) $m \angle d = 100$. Corollary 6-4-1.

5. $m \angle a = 55$.
   $m \angle b = 125$. Theorem 4-13.
   $m \angle c = 125$. Corollary 6-4-1.
   $m \angle d = 55$. Theorem 4-13.
   $m \angle e = 55$. Theorem 4-19 (or Theorem 4-13, or Corollary 6-4-1).
   $m \angle f = 55$. Theorem 6-4 (or Corollary 6-4-2, or any of several other satisfactory reasons).

6. (a) $\angle A$ and $\angle D$; $\angle B$ and $\angle C$.
   (b) $\angle C$ and $\angle D$; $\angle B$ and $\angle A$.

7. $\angle r$ and $\angle x$; $\angle s$ and $\angle t$.

8. Proof:

Statements	Reasons
1. $\triangle XYZ$ is isosceles.	1. Hypothesis.
2. $\angle Y \cong \angle Z$.	2. Base angles of an isosceles triangle are congruent.
3. $\overleftrightarrow{AB} \parallel \overleftrightarrow{YZ}$.	3. Hypothesis.
4. $\angle XAB \cong \angle Y$, $\angle XBA \cong \angle Z$.	4. Corollary 6-4-1.
5. $\angle XAB \cong \angle XBA$.	5. Transitive property of congruence for angles.
6. $\triangle XAB$ is isosceles.	6. If a triangle has two congruent angles, it is isosceles.

9. Proof:

Statements	Reasons
1. $\overrightarrow{PQ}$ bisects $\angle SPR$.	1. Hypothesis.
2. $\angle SPQ \cong \angle RPQ$.	2. Definition of bisect.
3. $\overleftrightarrow{RQ} \parallel \overleftrightarrow{PS}$.	3. Hypothesis.
4. $\angle RQP \cong \angle SPQ$.	4. Theorem 6-4.
5. $\angle RQP \cong \angle RPQ$.	5. Transitive property of congruence for angles.
6. $\triangle PRQ$ is isosceles.	6. If a triangle has two congruent angles, it is isosceles.

10. Proof:

Statements	Reasons
1. $RT = RS$.	1. Hypothesis.
2. $\angle T \cong \angle S$.	2. If two sides of a triangle are congruent, the angles opposite those sides are congruent.
3. $\overleftrightarrow{PQ} \parallel \overleftrightarrow{RS}$.	3. Hypothesis.
4. $\angle S \cong \angle TQP$.	4. Corollary 6-4-1.
5. $\angle T \cong \angle TQP$.	5. Property of transitivity for congruence of angles.
6. $\overline{PQ} \cong \overline{PT}$, or $PQ = PT$.	6. If two angles of a triangle are congruent, the sides opposite the congruent angles are congruent.

11. $x = 8$, since $3x + 14 = 5x - 2$.

(a) $m \angle BAC = 5x - 2 = 38$.

(b) $m \angle DCA = 3x + 14 = 38$.

(c) $m \angle ACB = 10x + 12 = 92$.

(d) $m \angle DCE = 50$.

(e) $m \angle CBA = m \angle DCE = 50$.

12. Proof:

Statements	Reasons
1. $\overrightarrow{RS}$ is the midray of $\angle ZRW$.	1. Hypothesis.
2. $\angle WRS \cong \angle ZRS$.	2. Definition of midray.
3. $\overleftrightarrow{RS} \parallel \overleftrightarrow{YZ}$.	3. Hypothesis.
4. $\angle ZRS \cong \angle YZR$.	4. Theorem 6-4.
5. $\angle ZYR \cong \angle WRS$.	5. Corollary 6-4-1.
6. $\angle ZYR \cong \angle YZR$.	6. Property of transitivity for congruence of angles.
7. $\triangle RYZ$ is isosceles.	7. If two angles of a triangle are congruent, the sides opposite the congruent angles are congruent.

13. Proof:

Statements	Reasons
1. W is the midpoint of $\overline{RX}$. S is the midpoint of $\overline{YR}$. S is the midpoint of $\overline{WZ}$.	1. Hypothesis.
2. $\overline{XW} \cong \overline{RW}$; $\overline{RS} \cong \overline{YS}$; $\overline{WS} \cong \overline{ZS}$.	2. Definition of a midpoint.
3. $\angle YSZ \cong \angle RSW$.	3. Any two vertical angles are congruent.
4. $\triangle WRS \cong \triangle ZYS$.	4. S.A.S.
5. $\overline{RW} \cong \overline{YZ}$, and $\angle WRS \cong \angle ZYS$.	5. Definition of congruence of triangles.
6. $\overleftrightarrow{YZ} \parallel \overleftrightarrow{XR}$.	6. Theorem 6-2.
7. $\overline{XW} \cong \overline{YZ}$.	7. Transitive property of congruence for segments.

14. $\triangle ABD \cong \triangle EDB$; $\triangle BCE \cong \triangle EDB$ by S.S.S. and $\angle ABD \cong \angle EDB$; $\angle CBE \cong \angle DEB$ by the definition of congruence. Therefore, $\overleftrightarrow{AB} \parallel \overleftrightarrow{DE}$; $\overleftrightarrow{BC} \parallel \overleftrightarrow{DE}$ by Theorem 6-2, and A, B, and C are collinear points by the Parallel Postulate.

Problem Set 6-6b

1. (a) Yes. Corollary 6-2-1.
   (b) Yes. Corollary 6-2-1.
   (c) Yes. Corollary 6-2-1 (or, Theorem 6-5) .

2.

Statements	Reasons
1. $\overleftrightarrow{XY} \parallel \overleftrightarrow{RS}$.	1. Hypothesis.
2. There is ray $\overrightarrow{PF}$ such that $\overleftrightarrow{PF} \parallel \overleftrightarrow{XY}$ and $\overrightarrow{PF}$ is between $\overrightarrow{PS}$ and $\overrightarrow{PY}$.	2. Theorem 6-3.
3. $\overleftrightarrow{PF} \parallel \overleftrightarrow{RS}$.	3. Theorem 6-5.
4. $\angle PSR \cong \angle SPF$, $\angle XYP \cong \angle FPY$.	4. Theorem 6-4.
5. $m \angle SPY = m \angle SPF + m \angle FPY$.	5. The Betweenness-Angles Theorem (Theorem 4-4).
6. $m \angle SPY = m \angle PSR + m \angle XYP$.	6. Substitution property of equality.

3. (a) Let ℓ_1 and ℓ_2 be distinct parallel lines, and let a line m in the plane of ℓ_1 and ℓ_2 intersect ℓ_1 at a point P . Suppose that m does not intersect ℓ_2 in a single point; then $m \parallel \ell_2$, by definition. But this contradicts the Parallel Postulate. Hence m must intersect ℓ_2 .

(b) Corollary 6-5-1 is essentially a contrapositive of the coplanar case of Theorem 6-5; for the superior student who recognizes this logical relationship, no further proof may be necessary. Many students may prefer the following proof.

Let ℓ_1 and ℓ_2 be distinct parallel lines, and let a line m in the plane of ℓ_1 and ℓ_2 intersect ℓ_1 in a point. Suppose that m does not intersect ℓ_2 in a single point; then $m \parallel \ell_2$, by definition. Consequently, by Theorem 6-5, $m \parallel \ell_1$. This is impossible. Hence m intersects ℓ_2.

(c) If r intersects only one of the lines ℓ_1 and ℓ_2 that are coplanar with it, then r is parallel to the other of them. The one that r intersects cannot be parallel to the other without contradicting the Parallel Postulate. Consequently the two lines ℓ_1 and ℓ_2 intersect.

4. Let p, q, r be three distinct coplanar lines such that $p \parallel r$ and $q \parallel r$. Let A be any point on r. There is a line u containing A and perpendicular to p. Since u is a transversal of p and r, Corollary 6-4-3 tells us that $u \perp r$. Similarly, there is a line v containing A and perpendicular to q; by Corollary 6-4-3; $v \perp r$. By Theorem 4-21, u and v are the same line. Since each of p and q is perpendicular to u, we apply Theorem 6-1 to conclude that $p \parallel q$.

5. Case (a): $s = s'$. If also $t = t'$ the result follows from the hypothesis. If $t \neq t'$, then s (where $s = s'$) is a transversal of t and t', and it is perpendicular to t', and hence perpendicular to t by Corollary 6-4-3.

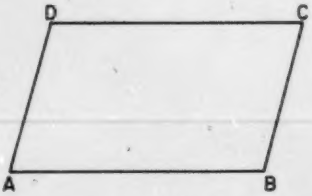
Case (b): $t = t'$. If also $s = s'$ the result follows from the hypothesis. If $s \neq s'$, then t (where $t = t'$) is a transversal of s and s' and it is perpendicular to s' , and hence to s .

Problem Set 6-7

Problems 9, 10, 11 of this section lead to the important summary with respect to parallelism and antiparallelism for the corresponding sides of two distinct angles in a plane. The proofs of these problems are time-consuming, and the teacher should exercise caution when making the assignment. Perhaps a desirable solution would be to ask students to prove 9(a), 10(a), 11(a), and 11(b). The proofs to 9(b), 10(b), and 11(c) could be handled more conveniently by blackboard demonstration and class discussion. This is especially true since it is necessary to use a general case for these problems rather than rely on a picture of a specific case for proof. For example, Problem 9(b) would require four different pictures if this approach is used.

1. The missing reasons are:

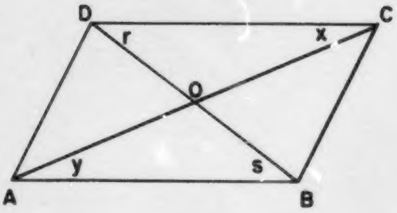
   2. Reflexive property of congruence for segments.
   4. A.S.A.
   5. Definition of congruence for triangles.



*2. If ABCD is a parallelogram, we are to prove that $\angle A \cong \angle C$ and that $\angle B \cong \angle D$. Since the parallel lines $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ and their transversal $\overleftrightarrow{AB}$ determine $\angle A$ and $\angle B$ as a pair of consecutive interior angles, $\angle A$ and $\angle B$ are supplementary, by Corollary 6-4-2. Likewise, $\angle B$ and $\angle C$ are supplementary, because $\overleftrightarrow{CD} \parallel \overleftrightarrow{AB}$. Therefore $\angle A$ and $\angle C$ are congruent, since each is a supplement of $\angle B$. The proof that $\angle B \cong \angle D$ is similar.

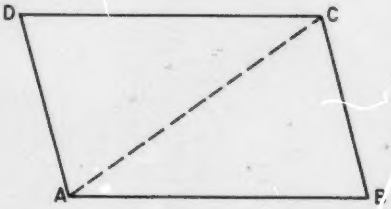
3. Hypothesis: ABCD is a parallelogram with diagonals $\overline{AC}$ and $\overline{BD}$ intersecting at O .

Prove: DO = OB and OA = OC .



Statements	Reasons
1. $\overline{DC} \parallel \overline{BA}$.	1. Definition of a parallelogram.
2. $\angle x \cong \angle y$ and $\angle r \cong \angle s$.	2. Theorem 6-4.
3. $\overline{DC} \cong \overline{BA}$.	3. In any parallelogram each side is congruent to the side opposite.
4. $\triangle DOC \cong \triangle BOA$.	4. A.S.A.
5. $\overline{DO} \cong \overline{OB}$ and $\overline{CO} \cong \overline{OA}$.	5. Definition of congruence for triangles.

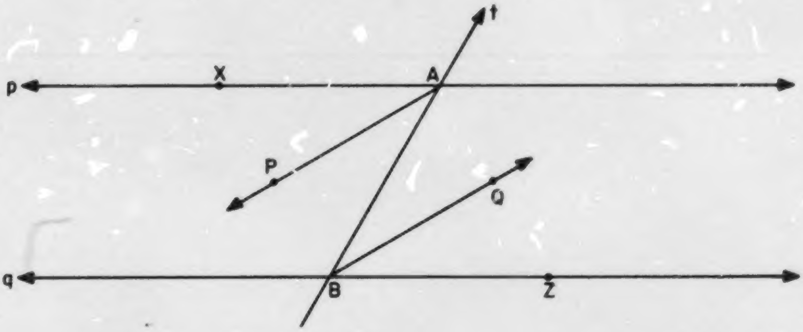
4. $\overline{AB} \cong \overline{CD}$ and $\overline{AB} \parallel \overline{CD}$ by hypothesis. $\angle BAC \cong \angle DCA$ by Theorem 6-4, and $\overline{AC} \cong \overline{CA}$ by the reflexive property of congruence for segments. Therefore, $\triangle BAC \cong \triangle DCA$ by S.A.S., and by the definition of congruence, $\angle BCA \cong \angle DAC$. Therefore, $\overline{BC} \parallel \overline{DA}$ by Theorem 6-2, and ABCD is a parallelogram by definition of a parallelogram.



*5. Use the same diagram as in Problem 4. $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{CB}$ by hypothesis. By the reflexive property of congruence for segments $\overline{AC} \cong \overline{CA}$. Therefore, $\triangle ACD \cong \triangle CAB$ by S.S.S. Hence, $\angle DAC \cong \angle BCA$ by definition of congruence. By Theorem 6-2, $\overline{BC} \parallel \overline{DA}$. Therefore ABCD is a parallelogram by Theorem 6-7.

6. $\overline{AC}$, $\overline{ZX}$, and $\overline{BY}$.

7. Proof: Given: $p \parallel q$; t intersects p and q at A and B, respectively. $\angle ZBA$ and $\angle XAB$ are a pair of alternate interior angles; $\overrightarrow{BQ}$ bisects $\angle ZBA$ and $\overrightarrow{AP}$ bisects $\angle XAB$. We are required to prove that $\overrightarrow{AP}$ and $\overrightarrow{BQ}$ are antiparallel.



	Statements		Reasons
1.	$p \parallel q$.	1.	Hypothesis.
2.	$\angle ZBA \cong \angle XAB$.	2.	Theorem 6-4.
3.	$\overrightarrow{AP}$ bisects $\angle XAB$, $\overrightarrow{BQ}$ bisects $\angle ZBA$.	3.	Hypothesis.
4.	$m\angle PAB = \frac{1}{2} m\angle XAB$, $m\angle QBA = \frac{1}{2} m\angle ZBA$.	4.	Definition of bisect.
5.	$m\angle PAB = m\angle QBA$.	5.	Multiplication property of equality.
6.	$\overleftrightarrow{AP} \parallel \overleftrightarrow{BQ}$.	6.	Theorem 6-2.
7.	$\overrightarrow{AP}$ and $\overrightarrow{BQ}$ are antiparallel.	7.	Definition of antiparallel rays.

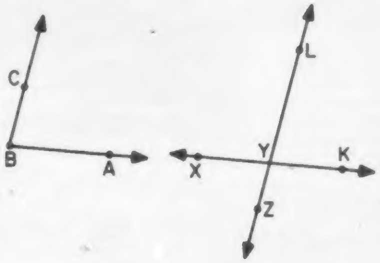
8.

	Statements		Reasons
1.	$\overrightarrow{AD}$ bisects $\angle CAB$.	1.	Hypothesis.
2.	$\angle BAR \cong \angle CAR$.	2.	Definition of bisect.
3.	$\overline{AR} \cong \overline{AR}$.	3.	Reflexive property for congruence of segments.
4.	$\overline{AD} \perp \overline{BC}$.	4.	Hypothesis.
5.	$\angle ARB \cong \angle ARC$.	5.	Two perpendicular segments determine right angles which are congruent
6.	$\triangle ARB \cong \triangle ARC$.	6.	A.S.A.
7.	$\angle ABR \cong \angle ACR$.	7.	Definition of congruence for triangles.
8.	$\overrightarrow{BC}$ bisects $\angle DBA$.	8.	Hypothesis.
9.	$\angle DBR \cong \angle ABR$.	9.	Definition of bisect.
10.	$\angle DBR = \angle ACR$.	10.	Transitive property for congruence of angles.
11.	$p \parallel q$.	11.	Theorem 6-2.

*9. (a) $\angle ABC \cong \angle XYZ$, by Corollary 6-4-1.

(b) Consider the lines $\overleftrightarrow{BA}$ and $\overleftrightarrow{YZ}$. These lines intersect at exactly one point, call it P . The point P is the endpoint of a unique ray which is parallel to $\overrightarrow{YX}$, call it $\overrightarrow{PQ}$. Also, P is the endpoint of a unique ray which is parallel to $\overrightarrow{BC}$, call it $\overrightarrow{PR}$. Since $\overrightarrow{BA}$ and $\overrightarrow{PQ}$ are collinear parallel rays and $\overrightarrow{BC}$ and $\overrightarrow{PR}$ are noncollinear parallel rays, $\angle ABC \cong \angle QPR$, by Part (a). Since $\overrightarrow{PR}$ and $\overrightarrow{YZ}$ are collinear parallel rays and $\overrightarrow{PQ}$ and $\overrightarrow{YX}$ are noncollinear parallel rays, $\angle QPR \cong \angle XYZ$ for the same reason. By the transitive property of congruence for angles, $\angle ABC \cong \angle XYZ$.

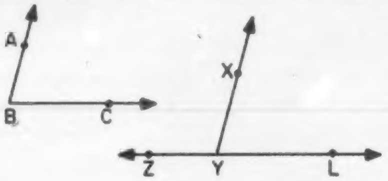
*10. Let $\overrightarrow{YK}$ be the ray opposite to $\overrightarrow{YX}$. Then, since $\overrightarrow{BA}$ and $\overrightarrow{YX}$ are antiparallel rays, $\overrightarrow{BA}$ and $\overrightarrow{YK}$ are parallel rays. Let $\overrightarrow{YL}$ be the ray opposite to $\overrightarrow{YZ}$. Then, since $\overrightarrow{BC}$ and $\overrightarrow{YZ}$ are antiparallel rays and are not collinear, $\overrightarrow{BC}$ and $\overrightarrow{YL}$ are parallel rays and are not collinear. Furthermore $\angle KYL \cong \angle XYZ$, because the angles are a pair of vertical angles.



(a) If $\overrightarrow{BA}$ and $\overrightarrow{YX}$ are collinear, then $\overrightarrow{BA}$ and $\overrightarrow{YK}$ are collinear. In this case, $\angle ABC \cong \angle KYL$ by Problem 9(a). By the transitive property of congruence for angles, $\angle ABC \cong \angle XYZ$.

(b) If $\overrightarrow{BA}$ and $\overrightarrow{YX}$ are not collinear, then $\overrightarrow{BA}$ and $\overrightarrow{YK}$ are not collinear. In this case, $\angle ABC \cong \angle KYL$ by Problem 9(b). By the transitive property of congruence for angles, $\angle ABC \cong \angle XYZ$.

*11. Let $\overrightarrow{YL}$ be the ray opposite to $\overrightarrow{YZ}$. Then $\overrightarrow{BC}$ and $\overrightarrow{YL}$ are parallel rays. Furthermore, $\angle XYL$ and $\angle XYZ$ are supplementary angles.



(a) Suppose that $\overrightarrow{BA}$ and $\overrightarrow{YX}$ are collinear and that $\overrightarrow{BC}$ and $\overrightarrow{YZ}$ are noncollinear. Then $\overrightarrow{BC}$ and $\overrightarrow{YL}$ are noncollinear. By Problem 9(a), $\angle ABC \cong \angle XYL$. Hence $\angle ABC$ and $\angle XYZ$ are supplementary.

(b) Suppose that $\overrightarrow{BA}$ and $\overrightarrow{YX}$ are noncollinear and that $\overrightarrow{BC}$ and $\overrightarrow{YZ}$ are collinear. Then $\overrightarrow{BC}$ and $\overrightarrow{YL}$ are collinear. Again we have the case treated in Problem 9(a), and $\angle ABC \cong \angle XYL$. As before, $\angle ABC$ and $\angle XYZ$ are supplementary.

(c) Suppose that $\overrightarrow{BA}$ and $\overrightarrow{YX}$ are noncollinear and that $\overrightarrow{BC}$ and $\overrightarrow{YZ}$ are noncollinear. Then $\overrightarrow{BC}$ and $\overrightarrow{YL}$ are noncollinear. By Problem 9(b), $\angle ABC \cong \angle XYL$. Therefore, $\angle ABC$ and $\angle XYZ$ are supplementary.

*12. (a) Congruent; parallel (or, antiparallel).
  (b) Congruent; antiparallel (or, parallel).
  (c) Supplementary; parallel (or, antiparallel); antiparallel (or, parallel).

Problem Set 6-8a

1. (a) 85 .
  (b) 1 .
  (c) 180-2n .
  (d) 180 - (r + s) .
  (e) 90 .
  (f) $90 - \frac{k}{2}$.

2. 36 .

3. 54 .

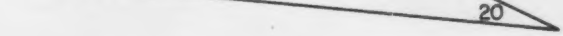
4. 36, 54, 90 .

5. 32, 96, 52; $x + 3x + (2x - 12) = 180$.

6. 4.2 .

7. The Parallel Postulate is used to prove Theorem 6-4 which is applied to obtain equations (2) and (3) in the proof of Theorem 6-9.

8.



9. Let $\angle DAC$ be an exterior angle of $\triangle ABC$. We are required to prove $m\angle DAC = m\angle B + m\angle C$.

Since $\angle DAC$ and $\angle CAB$ are a linear pair, they are supplementary and the sum of their measures is 180. By Theorem 6-9, $m\angle CAB + m\angle B + m\angle C = 180$. Therefore, $m\angle DAC = m\angle B + m\angle C$ by the addition property of equality.

10. (a) 120. (c) 155.
(b) 50. (d) 110.

11. 108.

12. 360.

13. Given $\triangle ABC$ and $\triangle XYZ$ with $m\angle A = m\angle X$ and $m\angle B = m\angle Y$. Then by the addition property of equality $m\angle A + m\angle B = m\angle X + m\angle Y$. Since $m\angle A + m\angle B + m\angle C = 180 = m\angle X + m\angle Y + m\angle Z$, then $m\angle C = m\angle Z$ by the addition property of equality.

14. Since $\overline{PD} \perp \overline{AB}$ and $\overline{PE} \perp \overline{AC}$, then $\angle BDP$ and $\angle CEP$ are right angles and congruent. $\angle B \cong \angle C$ since they are base angles of an isosceles triangle. Therefore, by Theorem 6-11, $\angle x \cong \angle y$.

15. $\angle AOB$ and $\angle DOC$ are congruent because any two vertical angles are congruent. $\angle B$ and $\angle C$ are right angles and therefore congruent since $\overline{AB} \perp \overline{BC}$ and $\overline{DC} \perp \overline{BC}$. Therefore, by Theorem 6-11, $\angle A \cong \angle D$.

16. (a) is correct by Theorem 6-11; (b) is not a valid conclusion because no information is given concerning any lengths, and therefore we cannot say that two lengths are the same.

17. Let the given triangles be $\triangle ABC$ and $\triangle A'B'C'$, with $\overline{AC} \cong \overline{A'C'}$, $\angle A \cong \angle A'$, and $\angle B \cong \angle B'$. By Theorem 6-11, $\angle C \cong \angle C'$. Therefore, $\triangle ABC \cong \triangle A'B'C'$ by A.S.A.

18. $\angle B$ and $\angle C$ are right angles and therefore congruent. $\angle BMA$ and $\angle CMD$ are vertical angles and therefore congruent. $\overline{AB} \cong \overline{DC}$. Hence, $\triangle ABM \cong \triangle DCM$ by S.A.A. By the definition of congruence $\overline{AM} \cong \overline{DM}$ and $\overline{BM} \cong \overline{CM}$. Therefore $\overline{AD}$ and $\overline{BC}$ bisect each other by the definition of bisect.

19. $\angle A$ and $\angle B$ are right angles and therefore congruent; by the definition of bisect, $\angle APD \cong \angle BPD$; $\overline{PD} \cong \overline{PD}$ by the reflexive property of congruence for segments. Therefore, $\triangle PDA \cong \triangle PDB$ by S.A.A. and by the definition of congruence, $\overline{AD} \cong \overline{BD}$.

Problem Set 6-8b

1. 60 .

2. 90 ; not necessarily. Although the sum of the measures of three angles of the quadrilateral is 270 , we do not know that the measure of each of the three angles is 90 .

3. The measure of each angle of the parallelogram is 90 . Use Corollary 6-4-3.

4. $m \angle C = 72$; $m \angle D = 108$; $m \angle B = 108$.
   Let $m \angle C = m \angle A = 72$. Let $m \angle B = m \angle D = x$.
   Therefore, $2x + 144 = 360$, and $x = 108$.

*5. By hypotheses ABCD is a quadrilateral with $\angle A \cong \angle C$, and $\angle B \cong \angle D$. We are required to prove ABCD is a parallelogram. Theorem 6-13 tells us that
$m \angle A + m \angle B + m \angle C + m \angle D = 360$.
$m \angle A + m \angle B + m \angle A + m \angle B = 360$ or
$2m \angle A + 2m \angle B = 360$ by the substitution property of equality. Therefore $m \angle A + m \angle B = 180$, and $\angle A$ and $\angle B$ are supplementary angles. By the substitution property of equality, $m \angle C + m \angle B = 180$; and $\angle C$ and $\angle B$ are supplementary angles. By Corollary 6-2-2, $\overline{AD} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{CD}$, and ABCD is a parallelogram.

Problem Set 6-9

1. 45 .

2. 18 .

3. Let $\triangle ABC$ be the given triangle with $m\angle A = 90$. By Theorem 6-9, $m\angle A + m\angle B + m\angle C = 180$. By the addition property of equality, $m\angle B + m\angle C = 90$. Therefore, $\angle B$ and $\angle C$ are complementary angles.

4. Since $\angle DAP$ and $\angle DBP$ are right angles, $\triangle DAP$ and $\triangle DBP$ are right triangles. By hypothesis, $\overline{AD} \cong \overline{BD}$. By the reflexive property of congruence for segments, $\overline{PD} \cong \overline{PD}$. Therefore, $\triangle DAP \cong \triangle DBP$ by Theorem 6-16 (the Hypotenuse-Leg Theorem). By the definition of congruence for triangles, $\angle APD \cong \angle BPD$. Thus $\overrightarrow{PD}$ is the midray of $\angle APB$.

5. By the Hypotenuse-Leg Theorem, $\triangle QTR \cong \triangle SVP$. Hence $\overline{QT} \cong \overline{SV}$. By the Betweenness-Addition Theorem, $\overline{QV} \cong \overline{ST}$. We can now apply S.A.S. to prove $\triangle PQV \cong \triangle RST$.

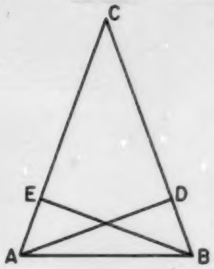
6. (a) By hypothesis, $AD = BC$. By Theorem 6-8, $AE = BF$. Since $\triangle ADE$ and $\triangle BCF$ are right triangles, $\triangle ADE \cong \triangle BCF$ by the Hypotenuse-Leg Theorem. Therefore, $\angle D \cong \angle C$. By Corollary 6-4-2, $\angle DAB$ and $\angle CBA$ are supplements of $\angle D$ and $\angle C$, respectively. Hence, $\angle DAB \cong \angle CBA$, because supplements of congruent angles are congruent.

   (b) By hypothesis, $\angle D \cong \angle C$. Also, $\angle DEA \cong \angle CFB$, since each of these angles is a right angle. By Theorem 6-8, $AE = BF$. Hence $\triangle DEA \cong \triangle CFB$ by S.A.A.; thus, $AD = BC$.

7. (a) Since $\overline{AD}$ is an altitude of $\triangle ABC$, the point D is contained in $\overleftrightarrow{BC}$. Since $\overrightarrow{AD} = \overrightarrow{AC}$, the point D is contained in $\overleftrightarrow{AC}$. Therefore $D = C$, because the lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{AC}$ intersect in exactly one point. Hence $\overline{AD} \cong \overline{AC}$.

With the aid of the hypotheses that $\overline{BE}$ is an altitude of $\triangle ABC$ and that $\overrightarrow{BE} = \overrightarrow{BC}$, we find similarly that $E = C$ and hence that $\overline{BC} \cong \overline{BE}$.

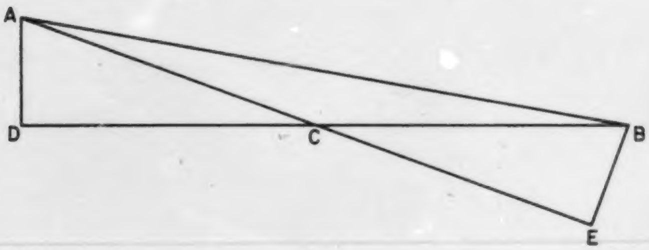
By hypothesis, $\overline{BE} \cong \overline{AD}$. Therefore by the transitive property of congruence for segments, $\overline{BC} \cong \overline{AC}$. Thus $\triangle ABC$ is isosceles. [An alternate approach for part of the proof: Since each of the points D and E is the same as C, the hypothesis $AD = BE$ becomes $AC = BC$ by the substitution property for equality; hence $\triangle ABC$ is isosceles.]



(b) Apply S.A.A. to prove $\triangle ACD \cong \triangle BCE$. Therefore $\overline{AC} \cong \overline{BC}$, and $\triangle ABC$ is isosceles by definition.

[Alternate proof: By the Hypotenuse-Leg Theorem, $\triangle ABE \cong \triangle BAD$, since $\overline{AB}$ is the hypotenuse for each of the right triangles. By the definition of congruence for triangles, $\angle EAB \cong \angle DBA$. Hence $\triangle ABC$ is isosceles by Theorem 5-7.

(c) Although the diagram for Part (c) is different from the diagram for Part (b), the proof for (b) is applicable here verbatim, excepting the reason for the statement that $\angle ACD \cong \angle BCE$.



Problem Set 6-10

1. (a) True.
   (b) False.

2. (a) True.
   (b) False.
   (c) True.
   (d) True.
   (e) False.
   (f) True.
   (g) False.

3. (a) False.
   (b) True.
   (c) False.
   (d) False.
   (e) True.
   (f) False.
   (g) True.

4. $x \leftrightarrow x'$, $y \leftrightarrow y'$, $z \leftrightarrow z'$;
   $x \leftrightarrow x'$, $y \leftrightarrow z'$, $z \leftrightarrow y'$;
   $x \leftrightarrow y'$, $y \leftrightarrow x'$, $z \leftrightarrow z'$;
   $x \leftrightarrow y'$, $y \leftrightarrow z'$, $z \leftrightarrow x'$;
   $x \leftrightarrow z'$, $y \leftrightarrow x'$, $z \leftrightarrow y'$;
   $x \leftrightarrow z'$, $y \leftrightarrow y'$, $z \leftrightarrow x'$.

5. (c) since $6 > 5 > 4$ and $75 > 60 > 45$.

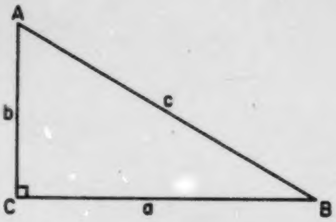
6. (f) .

Problem Set 6-11

1. Corollary 6-18-1.

We are given that $\angle C$ is a right angle of $\triangle ABC$. We are required to prove $c > a$ and $c > b$.

In right $\triangle ABC$, $\angle A$ and $\angle B$ are acute angles by Theorem 6-14. Therefore, $m \angle C > m \angle A$ and $m \angle C > m \angle B$. Then by Theorem 6-18, $c > a$ and $c > b$.



2. $\angle B$, $\angle C$, $\angle A$.

3. (a) $\overline{AC}$.
   (b) $\overline{BC}$.
   (c) $\overline{AB}$ ($m\angle C = 70$) .

4. (a) No.
   (b) No.
   (c) Yes.

5. 4 ; 20 .

6. 3 ; 13 .

7. Less than.

8. $j - k < x < j + k$.
   By Theorem 6-21, x must satisfy:
   $x < j + k$,
   $j < x + k$,
   $k < x + j$.

   Since $j > k$, these three conditions simplify to the statement that x is between $j - k$ and $j + k$.

9. $AH < AF$; $AH < AT$.
   $BT < TF$; $BT < AT$.
   Theorem 6-19 (or, Corollary 6-18-1) .

10. $HB < HC < HF$.
    Theorem 6-19 (or, Theorem 6-18); Theorem 6-18.

11. (a) $AD > AB$; Theorem 6-19 (or, Corollary 6-18-1).
    (b) $m\angle s > m\angle D$; Theorem 6-17 (or, Theorem 6-14).
    (c) $m\angle y > m\angle s$; Theorem 5-10.
    (d) $m\angle y > m\angle D$; Transitive property of order.
    (e) $AD > AE$; Theorem 6-18.
    (f) $AE = AC$; Definition of congruence ($\triangle ABE \cong \triangle ABC$ by S.A.S.).
    (g) $AD > AC$; Substitution property of equality.

12.

	Statements		Reasons
1.	$DB < CD + CB$, $DB < AD + AB$, $CA < CD + AD$, $CA < CB + AB$.	1.	The sum of the lengths of two sides of a triangle is greater than the length of the third side of the triangle.
2.	$2DB + 2CA < 2CD + 2AD + 2CB + 2AB$.	2.	Additive property of order.
3.	$DB + CA < CD + AD + CB + AB$.	3.	Multiplicative property of order.

Review Problems

1. Contrapositive of:

Theorem 6-4: If any two alternate interior angles determined by a transversal of two distinct lines are not congruent, then the lines are not parallel.

Corollary 6-4-1: If any two corresponding angles determined by a transversal of two distinct lines are not congruent, then the lines are not parallel.

Corollary 6-21-2: If any two consecutive interior angles determined by a transversal of two distinct lines are not supplementary, then the lines are not parallel.

Corollary 6-4-3: If a transversal is not perpendicular to one of two distinct parallel lines, then it is not perpendicular to the other.

Yes. By the Property of the Contrapositive.

2. $(3a + 15) + (2a - 35) = 180$.

Thus $a = 40$.

$p \parallel q \parallel s$.

3. (a) False.
(b) True.
(c) False.
(d) True.
(e) True.
(f) False.
(g) True.
(h) True.
(i) True.
(j) False.

4. $m\angle a = 105$.
$m\angle b = 80$.
$m\angle c = 80$.
$m\angle d = 100$.
$m\angle e = 25$.

5. $\angle B \cong \angle D$ because

(1) $\overline{AD}$ and $\overline{BC}$ are parallel and $\angle B$, $\angle D$ are alternate interior angles.

or (2) Considering vertical angles, if two angles of one triangle are congruent to two angles of another triangle, the third pair of angles are congruent.

6. (a) $m\angle a = 50$.
$m\angle b = 40$.
$m\angle c = 40$.
$m\angle d = 40$.
$m\angle e = 50$.
$m\angle x = 40$.
$m\angle y = 50$.

(b) $\overline{BG} \parallel \overline{DE}$ and $\overline{AB} \parallel \overline{DF}$.

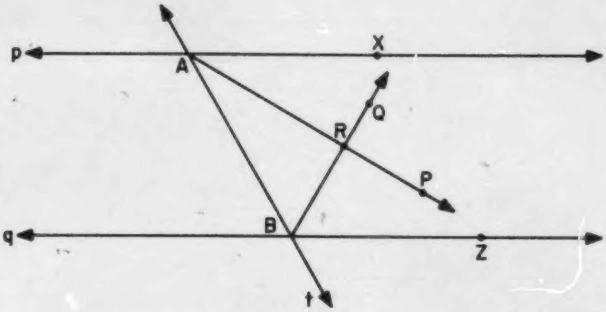
7. (a) 180 .
(b) 90 .
(c) 90 .
(d) 50 .
(e) b or a .

8. $CD + AD = AC$
$BE + EC = BC$
Thus $BE + EC + CD + AD = AC + BC$.
Since $\triangle ABC$ is isosceles, $\angle A \cong \angle B$.
Since $\overline{PE} \parallel \overline{AC}$ and $\overline{PD} \parallel \overline{BC}$, corresponding angles are congruent; that is, $\angle A \cong \angle EPB$, $\angle B \cong \angle DPA$.
Hence $\angle A \cong \angle DPA$ and $\angle B \cong \angle EPB$. Therefore, $AD = DP$ and $PE = EB$.

Substituting we see that

$$PE + EC + CD + DP = AC + BC .$$

9. Proof: Given $p \parallel q$; t intersects p and q at A and B , respectively, $\angle ZBA$ and $\angle XAB$ are a pair of consecutive interior angles. $\overrightarrow{AP}$ bisects $\angle XAB$ and $\overrightarrow{BQ}$ bisects $\angle ZBA$. We are required to prove $\overrightarrow{BQ} \perp \overrightarrow{AP}$ at R .

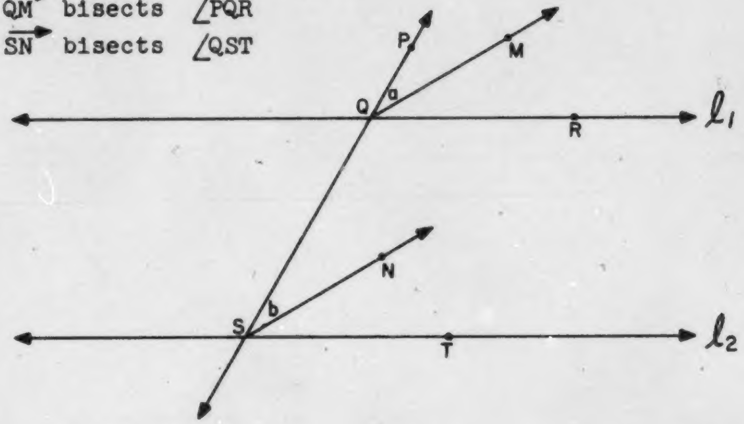


	Statements		Reasons
1.	$p \parallel q$.	1.	Hypothesis.
2.	$m\angle XAB + m\angle ZBA = 180$.	2.	Corollary 6-4-2.
3.	$\overrightarrow{AP}$ bisects $\angle XAB$, $\overrightarrow{BQ}$ bisects $\angle ZBA$.	3.	Hypothesis.
4.	$m\angle BAR = \frac{1}{2} m\angle XAB$, $m\angle ABR = \frac{1}{2} m\angle ZBA$.	4.	Definition of bisect
5.	$m\angle BAR + m\angle ABR = 90$.	5.	Addition and multiplication properties of equality.
6.	$m\angle BAR + m\angle ABR + m\angle ARB = 180$.	6.	Theorem 6-9.
7.	$m\angle ARB = 90$.	7.	Addition property of equality.
8.	$\overleftrightarrow{AP} \perp \overleftrightarrow{BQ}$.	8.	Definition of perpendicular lines.

10. Hypothesis: $\ell_1 \parallel \ell_2$

$\overrightarrow{QM}$ bisects $\angle PQR$

$\overrightarrow{SN}$ bisects $\angle QST$



$\angle PQR \cong \angle QST$, since $\ell_1 \parallel \ell_2$.

$m \angle a = \frac{1}{2} m \angle PQR$ and

$m \angle b = \frac{1}{2} m \angle QST$, by definition of bisect.

Hence $m \angle a = m \angle b$, by the multiplication property of equality. Therefore $\overleftrightarrow{QM} \parallel \overleftrightarrow{SN}$.

Finally, R and T lie on the same side of $\overleftrightarrow{QS}$, because $\angle PQR$ and $\angle QST$ are a pair of corresponding angles. Since M lies on the same side of $\overleftrightarrow{QS}$ as R does and N lies on the same side as T does, M and N are on the same side of $\overleftrightarrow{QS}$; in other words, $\overrightarrow{QM}$ and $\overrightarrow{SN}$ are parallel rays.

11. $\triangle BED \cong \triangle BEC$ by S.A.S.

Thus $ED = EC$.

Since $AC = AE + EC$,

$AC = AE + ED$.

Now, $AE + ED > AD$, by Theorem 6-21, since A, D, E are noncollinear and determine a triangle. Therefore, by the substitution property of equality,

$AC > AD$.

12. (a) Considering all possible cases:

Case (i) A, B, C are not three distinct points.

if (1) $A = B \neq C$, then $AB = 0$ and $AB + BC = AC$.

(2) $A = C \neq B$, then $AC = 0$ and $AB + BC > AC$.

(3) $A \neq B = C$, then $BC = 0$ and $AB + BC = AC$.

(4) $A = B = C$, then $AB = BC = AC = 0$ and $AB + BC = AC$.

Case (ii) A, B, C are three distinct collinear points.

(5) B is between A and C, then $AB + BC = AC$.

(6) A is between B and C, then $AB + AC = BC$ and $BC > AC$, thus $AB + BC > AC$.

(7) C is between A and B, then $AC + BC = AB$ and $AB > AC$, thus $AB + BC > AC$.

Case (iii) A, B, C are three noncollinear points. Then they determine a triangle ABC in which the sum of the measures of any two sides must be greater than the measure of the third, by Theorem 6-21. Thus $AB + BC > AC$.

(b)

Prove: If B is in $\overline{AC}$, then $AB + BC = AC$. Refer to Cases 1, 3, and 5 above in this problem, Part (a).

Conversely,

Prove: If A and C are distinct points and $AB + BC = AC$, then B must be in $\overline{AC}$.

Using the contrapositive of Theorem 6-21, we know that A, B, C cannot be noncollinear; therefore B is in $\overleftrightarrow{AC}$.

We must yet prove that B is in $\overline{AC}$.

We know that either

(1) C is between A and B ,
(2) A is between B and C , or
(3) B is in $\overline{AC}$.

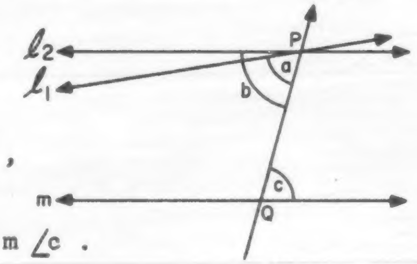
Case (1) is impossible since then $AB + BC > AC$ (see Case 7 in Part (a)).

Case (2) is impossible since then $AB + BC > AC$ (see Case 6 in Part (a)).

Thus Case (3) is the only remaining possibility and therefore B is in $\overline{AC}$.

13. The point G as described exists by the Point Plotting Theorem. $\triangle ABC \cong \triangle DGF$ by S.A.S. Thus $\angle DGF \cong \angle ABC$. It is given that $\angle ABC \cong \angle DEF$, therefore $\angle DGF \cong \angle DEF$. Now if $G \neq E$, then $\overleftrightarrow{FG} \parallel \overleftrightarrow{FE}$ because corresponding angles formed by the two lines and the transversal $\overleftrightarrow{GE}$ are congruent. But parallel lines cannot intersect in a single point. Therefore G and E cannot be distinct points; $G = E$. Thus $\triangle ABC \cong \triangle DEF$.

14. Line m and P determine a plane. P is on ℓ_1 and ℓ_2 . Since $\ell_1 \parallel m$ and $\ell_2 \parallel m$, ℓ_1 and ℓ_2 are coplanar with m and P . If $\ell_1 \parallel m$, then $m\angle a = m\angle c$. If $\ell_2 \parallel m$, then $m\angle b = m\angle c$.



Thus, $m\angle a = m\angle b$. Now both $\angle a$ and $\angle b$ have $\overrightarrow{PQ}$ as the zero-ray and the other sides of $\angle a$ and $\angle b$ extend into the same halfplane. (This is implied by the definition of alternate interior angles.) Therefore, by the Protractor Postulate $\ell_1 = \ell_2$. Thus, there is at most one line through P which is parallel to m .

15. $\ell_1 \parallel m$ and ℓ_1 contains P . There is a line t through P such that $t \perp m$. Thus ℓ_1 , m , and t are coplanar. Since ℓ_2 through P is parallel to m , ℓ_2 is coplanar with m , t , and ℓ_1 .

Now, since $t \perp m$, then $t \perp \ell_1$ and $t \perp \ell_2$ by Statement III. This tells us that in the same plane, ℓ_1 and ℓ_2 are perpendicular to a given line at a given point on the line. Theorem 4-21 tells us that there can be only one line in any one plane perpendicular to a given line in that plane at a given point on the line. Thus $\ell_1 = \ell_2$.

*16. A quadrilateral is a parallelogram if:

(a) each side is parallel to the side opposite it.
(b) each side is congruent to the side opposite it.
(c) each angle is congruent to the angle opposite it.
(d) two sides are parallel and congruent.
(e) the consecutive angles are supplementary.
(f) the diagonals bisect each other. (This will be proved in Chapter 8).

Chapter 7

ANSWERS AND SOLUTIONS

Problem Set 7-2a

1. (b) True $k = \frac{1}{3}$

(c) True $k = \frac{1}{6}$

(d) True $k = 2$

(e) True $k = 4$

(f) False

(g) True $k = -\frac{2}{3}$

(h) False

(i) True $k = \frac{1}{2}$

(j) True $k = \frac{1}{3}$

(k) True $k = \frac{1}{6}$

(l) True $k = 1$

(m) False

(n) False k cannot be 0 .

(o) True $k = -1$.

2. (a) 48, 9, 15 $k = 3$

(b) 3, 4 $k = \frac{1}{5}$

(c) 36, 72, 0, -18

(d) $\frac{1}{9}$ $k = 3$

(e) 10, 24, 26

3. All are correct conclusions.

*4. Yes. Yes.

*5. $\frac{1}{2}$. 2 . One is the multiplicative inverse (or reciprocal) of the other. Their product is one. This is an instance of the symmetric property of proportionality which is stated in the next section.

6. Yes. $(3, 5) \underset{p}{=} (9, 15) \quad k = \frac{1}{3}$,

$(9, 15) \underset{p}{=} (18, 30) \quad k = \frac{1}{2}$,

$(3, 5) \underset{p}{=} (18, 30) \quad k = \frac{1}{6} = \frac{1}{3} \cdot \frac{1}{2}$.

This is an instance of the transitive property of proportionality which is stated in the next section.

7. (a) Yes. By definition of proportionality.
(b) Yes. The quotient is 3 .
(c) If $6 = 3x$, then $6y = 3xy$ and
if $12 = 3y$, then $12x = 3xy$, by the
multiplication property of equality.
Thus $6y = 12x$ by the transitive property
of equality.

8. Yes. $d = k \cdot b$. Thus, $d = k \cdot 0$, and by a basic arithmetic property, if 0 is a factor, the product is 0 .

No. If $a \neq 0$, $c \neq 0$ because $c = k \cdot a$, and k cannot be 0 since c is not zero.

*9. Yes. Yes. By hypothesis, there is $k \neq 0$ such that $c = ka$ and $3 = 6k$. Hence, $k = \frac{1}{2}$ and $c = \frac{1}{2}a$. Therefore $\frac{a}{c} = 2 = \frac{6}{3}$ and

$$\frac{c}{3} = \frac{\frac{1}{2}a}{3} = \frac{a}{6} .$$

*10. Yes. Yes. By hypothesis, there is $k \neq 0$ such that $a = 2k$, $b = 3k$. Hence, $a = \frac{2}{3}b$. Since we also know that $2 = \frac{2}{3} \cdot 3$, $(a, 2) \underset{p}{=} (b, 3)$. Also, since $3 = \frac{3}{2} \cdot 2$ and $b = \frac{3}{2}a$, $(3, b) \underset{p}{=} (2, a)$

*11. (a) The correspondence indicated in $(a, b) \underset{p}{=} (c, d)$ is the same as that indicated in $(b, a) \underset{p}{=} (d, c)$ Hence, by definition, if one holds then the other does also.

(b) There is $q \neq 0$ such that $a = qb$, since a and b are not both zero. From the hypothesis $(a, b) \underset{p}{=} (c, d)$, there is $k \neq 0$ such that $a = kc$ and $b = kd$. Substituting these expressions for a and b in $a = qb$, we obtain $kc = qkd$. Since $k \neq 0$, we have $c = qd$. Since $a = qb$, $c = qd$, and $q \neq 0$, we have $(a, c) \underset{p}{=} (b, d)$.

(c) Since $(a, b) \underset{p}{=} (c, d)$, there is $k \neq 0$ such that $a = kc$ and $b = kd$. Hence $ad = (kc)d = (kd)c = bc$.

(d) Since a and c are not zero, there is k such that $a = kc$. Substituting in $ad = bc$, we obtain $(kc)d = bc$, that is, $b = kd$. Hence $(a, b) \underset{p}{=} (c, d)$.

With regard to the note, if $c = 0$, then $a = 0$ and $(0, 0) \underset{p}{=} (b, d)$ only if both b and d are zero. Thus, if $c = 0$ and $d \neq 0$, the conclusion in Part b would be false. Similarly, the conclusion in Part d would be false unless both $a = b = 0$ and $c = d = 0$ hold.

12. Yes, since if $a = kc$ and $b = kd$, then $5a = k(5c)$ and $b = kd$. The proportionality constants are the same.

13. (a) 8
(b) 6
(c) 3
(d) $\frac{9}{2}, \frac{9}{4}, \frac{9}{6}$
(e) $\sqrt{2}, 2\sqrt{2}, 4$
(f) $a, .3b, .124c$

(g) $\sqrt{3}, 3, 3\sqrt{3}, 9\sqrt{3}$

(h) 12

(i) $5\frac{1}{3}$

(j) $\sqrt{10}$

(k) $3\sqrt{6}$

(l) $(\underline{3\sqrt{2}}, 8, 6, \sqrt{2}) \underset{p}{=} (3, 4\sqrt{2}, \underline{3\sqrt{2}}, \underline{1})$

Problem Set 7-2b

1. (a) True. Reflexive property.

(b) True. Since $(x, y) \underset{p}{=} (r, s)$ and r, s, x, y are positive, $(y, x) \underset{p}{=} (s, r)$ by inversion.

Hence $(s, r) \underset{p}{=} (y, x)$ by the symmetric property.

Further, since $x = \frac{3}{2}r$, $r = \frac{2}{3}x$.

So the proportionality constant is $\frac{2}{3}$.

(c) False. $(x, y) \underset{p}{=} (2, 3)$.

(d) False. $(r, s) \underset{p}{=} (b, c)$ with $k = 3$.

(e) True. Symmetric property and inversion.

Since $2 = 3a$, $a = \frac{1}{3} \cdot 2$. Hence, $k = \frac{1}{3}$ for the proportion in the conclusion.

(f) True. By the addition property,
$(5, 2, 3) \underset{p}{=} (a + b, a, b)$. Hence,

$(2, 3, 5) \underset{p}{=} (a, b, a + b)$ since this is the same correspondence.

(g) True. Inversion

(h) False. $3x = 2y$

2. (a) $(7, 2) \underset{p}{=} (y, x)$

(b) $(z, x) \underset{p}{=} (8, 2)$

(c) $(10, 7) \underset{p}{=} (x + z, y)$

(d) $(6, 7) \underset{p}{=} (z - x, y)$

We have given only one proportion for each part. Students may use various acceptable proportions.

3. $(5, 2) \underset{p}{=} (y, x)$, $(5, y) \underset{p}{=} (2, x)$, $(5, 7) \underset{p}{=} (y, y + x)$.

The last of these is obtained from the first by addition which gives $(7, 5, 2) \underset{p}{=} (y + x, y, x)$.

Hence $(7, 5) \underset{p}{=} (y + x, y)$. So, by inversion,

$(5, 7) \underset{p}{=} (y, y + x)$.

4. (a) 4 (e) 4

(b) $6\frac{2}{3}$ (f) 6

(c) $1\frac{1}{3}$ (g) 3

(d) 9 (h) $\sqrt{2}$

5. $CE = 4\frac{1}{2}$ since by hypothesis $(AD, DB) \underset{p}{=} (CE, EB)$ and so $(3, 4) \underset{p}{=} (CE, 6)$; that is, $18 = 4 \cdot CE$.

6.

1. $(AD, DB) \underset{p}{=} (CE, EB)$	1. Hypothesis
2. $(AD, DB, AD + DB) \underset{p}{=} (CE, EB, CE + EB)$	2. Addition property of proportionality
3. $AD + DB = AB$ and $CE + EB = CB$	3. Betweenness-Distance Theorem
4. $(AD, DB, AB) \underset{p}{=} (CE, EB, CB)$	4. Substitution property of equality for real numbers

7. $XY = a + b = 6$ and $a = 2$, hence $b = 6$.
$ZW = c + d = 8$. Since $(a, b) \underset{p}{\equiv} (c, d)$, we have
$(a, b, a + b) \underset{p}{\equiv} (c, d, c + d)$.

That is, $(2, 4, 6) \underset{p}{\equiv} (c, d, 8)$ with $k = \frac{3}{4}$.

$$2 = \tfrac{3}{4}c \ , \ 4 = \tfrac{3}{4}d$$

$$c = \frac{8}{3} \ , \ d = \frac{16}{3}$$

8. $d = 4\frac{1}{2}$, $e = 6$

Problem Set 7-3

1. 4', 2', 24', $\frac{1}{2}$'.
3" x 5".

2. (a) $k = \frac{1}{3}$ $DF = 21$ $FE = 18$
(b) $k = 2$ $MN = 9$, $MP = 8$
(c) $k = \frac{3}{2}$ $BC = 12$, $AE = 12$
(d) $k = 2$ $YZ = 8$, $XZ = 4\sqrt{3}$
(e) $k = 3$ $RT = 12$, $ST = 15$
(f) $k = \sqrt{2}$ $AC = \sqrt{2}$, $BC = 2$
(g) $k = 3$ $MN = 3$, $NP = 3\sqrt{3}$

3. (a) $BC = 4$, $AC = 2\sqrt{3}$, $RS = \sqrt{3}$, $ST = 2\sqrt{3}$
(b) (1) 2
(2) $\frac{1}{2}$
(3) $\frac{2}{3}\sqrt{3}$
(4) $\frac{\sqrt{3}}{3}$ or $\frac{1}{\sqrt{3}}$
(5) 1

4. 1

5. 8

6. $\frac{1}{\sqrt{3}}$ or $\frac{\sqrt{3}}{3}$

7. $k = 2 , \frac{2}{1} , \frac{2}{1}$

8. 27, 36, 33 . Corresponding angles are congruent.

9. 33 . They are congruent.

10. Hypothesis: $\triangle ABC \sim \triangle DEF$, $\triangle DEF \sim \triangle XYZ$.

Therefore $(AB, BC, AC) \underset{p}{=} (DE, EF, DF)$

$(DE, EF, DF) \underset{p}{=} (XY, YZ, XZ)$ by the definition of similarity.

Thus $(AB, BC, AC) \underset{p}{=} (XY, YZ, XZ)$ by the transitive property of proportionality.

Also $\angle A \cong \angle D$ and $\angle D \cong \angle X$,
$\angle B \cong \angle E$ and $\angle E \cong \angle Y$,
$\angle C \cong \angle F$ and $\angle F \cong \angle Z$.

Thus $\angle A \cong \angle X$, $\angle B \cong \angle Y$ and $\angle C \cong \angle Z$ by the transitive property of congruence for angles.

Therefore $\triangle ABC \sim \triangle XYZ$ since this correspondence has corresponding angles congruent and corresponding sides proportional.

11. $\triangle ABC \sim \triangle MNP$.

12. Yes. We know $(AB, BC, AC) \underset{p}{=} (XY, YZ, XZ)$ and $(XY, YZ, XZ) \underset{p}{=} (ZY, YX, ZX)$. In the second proportionality, $k = 1$, since $XZ = ZX$. Hence, $XY = YZ$. Therefore, from the first proportionality, $AB = BC$. Since $AB = 8$, this yields $BC = 8$.

13. Yes, since the constant of proportionality is 1 .

14. (a) $\triangle ABC \sim \triangle EDC$
(b) $\triangle ACD \sim \triangle CBD$
(c) $\triangle RST \sim \triangle RWV$
(d) $\triangle XZW \sim \triangle XYR$
(e) $\triangle MNP \sim \triangle RSQ$
(f) $\triangle ABC \sim \triangle EDF$
(g) $\triangle ADC \sim \triangle FBE$

Problem Set 7-4

1. (a) 11.2 . By Postulate 21,
(AB, AD, DB) $\underset{p}{=}$ (AC, AE, EC) . Hence
(10, 7, DB) $\underset{p}{=}$ (16, AE, EC) . Thus
(10, 7) $\underset{p}{=}$ (16, AE) , so 10AE = (7)(16) .

(b) (AD, DB, AB) $\underset{p}{=}$ (AE, EC, AC)

(1) $(3, 3, \underline{6}) \underset{p}{=} (4, \underline{4}, \underline{8})$

(2) $(\underline{4}, 2, 6) \underset{p}{=} (\underline{8}, 4, \underline{12})$

(3) $(3, \underline{4}, 7) \underset{p}{=} (\underline{3\frac{3}{4}}, 5, \underline{8\frac{3}{4}})$

(4) $(2, 3, \underline{5}) \underset{p}{=} (\underline{2\frac{4}{5}}, \underline{4\frac{1}{5}}, 7)$

(5) $(2, 1, \underline{3}) \underset{p}{=} (\underline{4}, 2, \underline{6})$

(c) No. Since (AD, AB) $\underset{p}{=}$ (AE, AC) , then by alternation, (AD, AE) $\underset{p}{=}$ (AB, AC) .

2. (a) b
(b) x + y
(c) b
(d) y, y
(e) a
(f) b

3. (a) AB = $5\frac{5}{7}$

(b) BF = 5

(c) BF = $13\frac{1}{2}$

4. In $\triangle$ACF , (AB, BC, AC) $\underset{p}{=}$ (AX, XF, AF) .

In $\triangle$AFD , (AX, XF, AF) $\underset{p}{=}$ (DE, EF, DF) .

By the transitive property of proportionality, (AB, BC, AC) $\underset{p}{=}$ (DE, EF, DF) .

5. Consider $\overleftrightarrow{BF} \parallel \overleftrightarrow{DE}$ and intersecting $\overleftrightarrow{AC}$ at F . Then, by Postulate 21, $(AB, AD) \underset{p}{=} (AF, AE)$. Thus, $AC = AF$. F is on $\overrightarrow{AC}$ since it is in the halfplane determined by $\overleftrightarrow{DE}$ and B . Thus, C and F must be the same point by the Point-Plotting Theorem. Therefore $\overleftrightarrow{BF}$ and $\overleftrightarrow{BC}$ are the same line since only one line can contain two points. Consequently $\overline{BC} \parallel \overline{DE}$.

Converse: If a line intersects two sides of a triangle in interior points so that the measures of one of those sides and the two segments into which it is cut are proportional to the measures of the three corresponding segments in the other side, then the line is parallel to the third side of the triangle.

6. $PQ = 7\frac{1}{2}$, $CD = 2\frac{2}{3}$.

7. $DF = 8\frac{1}{3}$

8. $CD = 3$, $OY = 5\frac{1}{4}$

9. Since $\overleftrightarrow{FG} \parallel \overleftrightarrow{AC}$, $\angle BFG \cong \angle A$.
Since $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$, $\angle ADE \cong \angle B$.
$\overline{BF} \cong \overline{AD}$. Therefore $\triangle ADE \cong \triangle FBG$ by A.S.A.

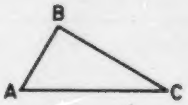
10. (a) $AD = 6$.
(b) $AE = \frac{2}{3}AC$. Postulate 21. $AE = 8$.
(c) Yes.
(d) $BG = \frac{2}{3}BC$. Postulate 21. $BG = 10$.
(e) Yes. This was proved in Problem 9 by A.S.A.
(f) Yes. $DE = 10$.
(g) Yes, with $k = \frac{2}{3}$.
(h) Corresponding angles must be proved congruent. $\angle BAC \cong \angle DAE$. Reflexive property of congruence. $\angle ABC \cong \angle ADE$ and $\angle ACB \cong \angle AED$ since $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$. Thus, $\triangle ADE \sim \triangle ABC$.

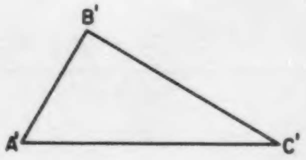
11. Let P' and Q' denote the points on $\overline{AB}$ that are also on the lines parallel to $\overleftrightarrow{AB}$ and through P and Q , respectively. By Theorem 7-2, since PQ = QR , P'Q' = Q'B . By Postulate 21, since AP = PQ , AP' = P'Q' . Hence AP' = P'Q' = Q'B , that is, the parallel lines trisect $\overline{AB}$.

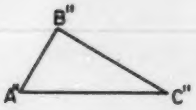
Problem Set 7-5

1. (a) S.S.S. Th. 7-4 , $\triangle ABC \sim \triangle DFE$
   (b) S.A.S. Th. 7-5 , $\triangle GHJ \sim \triangle MKL$
   (c) S.A.S. Th. 7-5 , $\triangle QRN \sim \triangle SRT$
   (d) A.A. Th. 7-6 , $\triangle ABC \sim \triangle FDE$
   (e) A.A. Th. 7-6 , $\triangle ABE \sim \triangle DCE$
   (f) S.A.S. Th. 7-5 , $\triangle MKN \sim \triangle LKP$
   (g) S.S.S. Th. 7-4 , $\triangle ABC \sim \triangle DFE$
   (h) A.A. Th. 7-6 , $\triangle ACD \sim \triangle CBD$ (Also $\triangle ACD \sim \triangle ABC$ and $\triangle CBD \sim \triangle ABC$)
   (i) S.A.S. Th. 7-5 , $\triangle RST \sim \triangle WSV$

2.







1. $AB = kA'B'$, $BC = kB'C'$ $\angle B \cong \angle B'$	1. Hypothesis
2. $\triangle A''B''C'' \sim \triangle A'B'C'$ with proportionality constant k	2. Hypothesis and Theorem 7-3
3. $A''B'' = kA'B'$ $\angle B'' \cong \angle B'$ $B''C'' = kB'C'$	3. Definition of polygon similarity
4. $AB = A''B''$ $BC = B''C''$ $\angle B \cong \angle B''$	4. Transitivity for congruence and equality
5. $\triangle ABC \cong \triangle A''B''C''$	5. S.A.S. Postulate
6. $\triangle ABC \sim \triangle A'B'C'$	6. Transitivity for similarity

3. Use figure for solution of Problem 2 and the hint in the text following the statement of the theorem.

1. $\angle A \cong \angle A'$, $\angle B \cong \angle B'$	1. Hypothesis
2. $AB = kA'B'$	2. If a , b are positive, then there is a positive number k such that $a = kb$.
3. $\triangle A''B''C'' \sim \triangle A'B'C'$ with k as the proportionality constant	3. Theorem 7-3 asserts the existence of such a triangle.
4. $\angle A'' \cong \angle A'$ $\angle B'' \cong \angle B'$ $A''B'' = kA'B'$	4. Definition of similarity
5. $\angle A \cong \angle A''$ $\angle B \cong \angle B''$	5. Transitivity for congruence
6. $\overline{AB} \cong \overline{A''B''}$	6. Segments with equal measures are congruent.
7. $\triangle ABC \cong \triangle A''B''C''$	7. A.S.A. Postulate
8. $\triangle ABC \sim \triangle A'B'C'$	8. Transitivity for similarity

4. (a) 14

   (b) 12

   (c) 16

   (d) Insufficient information

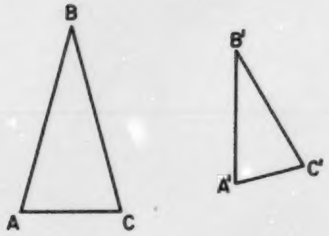
   (e) 6

   (f) $2\frac{2}{3}$

5. (a) If two lines are parallel and cut by a transversal, the corresponding angles are congruent, $\angle CBD \cong \angle A$, $\angle CDB \cong \angle E$.
   $\triangle ACE \sim \triangle BCD$ by A.A. Similarity Theorem.

   (b) $\angle CBD \cong \angle A$ by definition of polygon similarity. Thus $\overline{BD} \parallel \overline{AE}$ since, if two lines are cut by a transversal so that a pair of corresponding angles are congruent, the lines are parallel.

   (c) $BC = \frac{1}{2}AC$, $CD = \frac{1}{2}CE$ by definition of midpoint.
   $\angle C \cong \angle C$. Reflexive property for congruence.
   $\triangle ACE \sim \triangle BCD$ by S.A.S. Similarity Theorem and the constant of proportionality is 2 .

6. 

Since the triangles ABC and A'B'C' are isosceles, $\angle A \cong \angle C$, $\angle A' \cong \angle C'$. If $\angle A \cong \angle A'$, then $\angle C \cong \angle C'$ by the transitive property for congruence. Thus $\triangle ABC \sim \triangle A'B'C'$ by A.A. Similarity Theorem.

7. Given $\triangle ABC$ and $\triangle A'B'C'$ are isosceles with $\angle B \cong \angle B'$, the vertex angles. We know that $m\angle A = \frac{180 - m\angle B}{2}$ and $m\angle A' = \frac{180 - m\angle B'}{2}$ since base angles of isosceles triangles are congruent and the sum of the angles of a triangle is 180 . Also, $m\angle B = m\angle B'$, since congruent angles have equal measures. Thus $m\angle A = m\angle A'$ and $\angle A \cong \angle A'$. Therefore $\triangle ABC \sim \triangle A'B'C'$ by A.A. Similarity Theorem.

8. $\angle QTP \cong \angle QRS$ since they are right angles. $\angle Q \cong \angle Q$ by reflexive property for congruence. Therefore, $\triangle QTP \sim \triangle QRS$ by A.A. Similarity Theorem and $(QP, PT, TQ) \underset{p}{=} (QS, SR, RQ)$.

9. (a) $\angle BDE \cong \angle CFE$ since they are right angles. $\angle B \cong \angle C$ since they are opposite congruent sides of $\triangle ABC$. $\triangle BDE \sim \triangle CFE$ by A.A. Similarity Theorem.

   (b) Since $\triangle BDE \sim \triangle CFE$ the corresponding sides are proportional by the definition of polygon similarity. Hence $(BD, DE, EB) \underset{p}{=} (CF, FE, EC)$.

   (c) From $(BD, DE, EB) \underset{p}{=} (CF, FE, EC)$ we get $(DE, EB) \underset{p}{=} (FE, EC)$. Applying the alternation property of proportions, we get $(DE, FE) \underset{p}{=} (EB, EC)$.

10. (a) $\angle BZC \cong \angle AYC$ since they are right angles. $\angle C \cong \angle C$ by the reflexive property for congruence. Thus $\triangle BZC \sim \triangle AYC$ by A.A. Similarity Theorem. Therefore $(BC, BZ) \underset{p}{=} (AC, AY)$ by the definition of polygon similarity and thus by alternation, $(BC, AC) \underset{p}{=} (BZ, AY)$.

    (b) By the product property, $(BC)(AY) = (AC)(BZ)$.

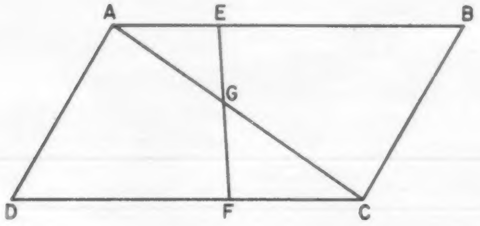
11. $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$ by definition of parallelogram. Thus $\angle BAE \cong \angle DFE$ and $\angle ABE \cong \angle FDE$. $\triangle FDE \sim \triangle ABE$ by A.A. Similarity Theorem, with $(DE, FD) \underset{p}{=} (BE, AB)$. If $DE = \frac{1}{2}EB$, $FD = \frac{1}{2}AB$ by the definition of proportionality. Since opposite sides of a parallelogram are congruent, $AB = DC$ and thus $FD = \frac{1}{2}DC$.

12. 39' is the height of the tree.

13. (a) The triangles are similar by S.S.S. Similarity Theorem, thus corresponding angles are congruent. If one has a right angle so does the other.

(b) 3k , 4k , 5k .

14.



$\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$, thus $\angle BAC \cong \angle DCA$ and $\angle AEG \cong \angle CFG$. Therefore $\triangle AEG \sim \triangle CFG$ by A.A. Similarity Theorem. Thus $(AG, GE) \underset{p}{=} (CG, FG)$. By the product property of proportion, $(AG)(GF) = (GE)(CG)$ and since multiplication is commutative, $(AG)(GF) = (CG)(GE)$.

15. (a) Approximately 2100 miles.

(b) Approximately 830,000 miles.

Problem Set 7-6a

2. $A'P' = 3$. $P'B' = 6$.

3. (a) (1) No

(2) Yes

(3) Yes

(b) No. Not if the segment is perpendicular to the line or in a plane perpendicular to the line.

(c) $PQ > P'Q'$.

4. $\overline{AD}$; $\overline{DB}$. AB

5. (a) $m\angle ACD = 50$

$m\angle DCB = 40$

$m\angle CBD = 50$

(b) $\angle A \cong \angle DCB$. $\angle B \cong \angle ACD$.

(c) Yes

(d) 3

(e) $\triangle ACD \sim \triangle CBD \sim \triangle ABC$

Problem Set 7-6b

The teacher may want to ask students to prove Theorem 7-7 in a two-column proof as an additional problem.

1.

$\triangle ACB$	$\triangle CDB$	$\triangle ADC$
c	a	b
a	x	h
b	h	y

2. 7-7-1 $h^2 = xy$

7-7-2 $a^2 = xc$, $b^2 = yc$

3. (a) Since $\triangle CDB \sim \triangle ADC$, $(x, h) \underset{p}{=} (h, y)$.

By the product property of proportions,

$$h^2 = xy \ .$$

(b) Since $\triangle ACB \sim \triangle CDB$, $(c, a) \underset{p}{=} (a, x)$,

thus $a^2 = xc$.

Since $\triangle ACB \sim \triangle ADC$, $(c, b) \underset{p}{=} (b, y)$,

thus $b^2 = yc$.

4. Since $\triangle ACB \sim \triangle ADC$, $(c, a) \underset{p}{=} (b, h)$,

and thus $ch = ab$.

Multiplying by $\frac{1}{c}$, $h = \frac{ab}{c}$.

5. (a) 4
(b) 3
(c) 7
(d) 8.5
(e) 2
(f) 6
(g) 6
(h) 2
(i) 4
(j) 5 or 20

Problem Set 7-7

1. c, f, g, i, k are not right triangles because the square of any one side is not the sum of the squares of the other two sides.

   a, b, d, e, h, j, l, m, n, o are right triangles because the square of one side is the sum of the squares of the other two sides.

2. (a) 16

   (b) $\sqrt{130}$

   (c) $10\sqrt{2}$

   (d) $2\sqrt{39}$

   (e) $2\frac{1}{2}$

3. $3\sqrt{11}$

4. 75

5. 7

6. $2\sqrt{73}$

7. $\frac{8}{\sqrt{3}}$ or $\frac{8}{3}\sqrt{3}$

8. 20

9. 12

10. 10

11. 13

12. $\sqrt{2}$

13. 6

14. x , $x\sqrt{3}$

15. There is a right triangle whose legs have lengths u and v (see proof preceding the problem set) by the Ruler Postulate, Protractor Postulate, and Point-Plotting Theorem. Then, by the Pythagorean Theorem, the hypotenuse x is such that $x^2 = u^2 + v^2$. Therefore $x^2 = w^2$ and since both x and w are positive, $x = w$.

16. $(x^2 - 1)^2 + (2x)^2 = x^4 - 2x^2 + 1 + 4x^2 = x^4 + 2x^2 + 1$

$$= (x^2 + 1)^2 \ .$$

Since the sum of the squares of two sides is the square of the third, the triangle is a right triangle.

If $x = 2$, the sides are $3, 4, 5$.
If $x = 4$, the sides are $15, 8, 17$.
If $x = 6$, the sides are $35, 12, 37$.

17. $(2uv)^2 + (u^2 + v^2)^2 = 4u^2v^2 + u^4 - 2u^2v^2 + v^4$

$$= u^4 + 2u^2v^2 + v^4$$

$$= (u^2 + v^2)^2 \ .$$

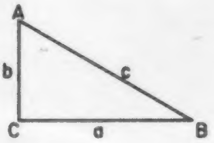
Since the sum of the squares of two sides is the square of the third, it is a right triangle.

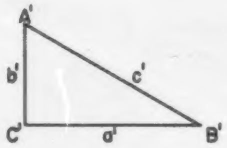
If $u = 5$ and $v = 2$, these numbers are $20, 21, 29$.

18. Either use the formulas in 16 and 17 or multiply the integers 29, 20, 21 by 1000 to obtain the triple 29,000 , 20,000 , 21,000 .

19. Yes. S.A.S. Similarity Theorem.

20. Yes





It is given that

$a = ka'$, $c = kc'$

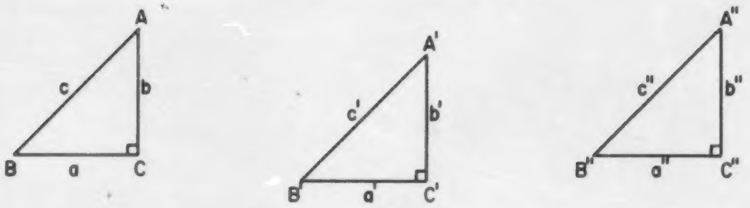
$b = \sqrt{(kc')^2 - (ka')^2} = \sqrt{k^2(c'^2 - a'^2)} = k\sqrt{c'^2 - a'^2}$

$b' = \sqrt{c'^2 - a'^2}$.

Therefore $b = kb'$.

Thus $\triangle ACB \sim \triangle A'C'B'$ by S.S.S. Similarity Theorem.

Alternate proof:



Hypothesis: $\triangle ABC$, $\triangle A'B'C'$ have right angles at C, C'.

$a = ka'$, $c = kc'$.

Prove: $\triangle ABC \sim \triangle A'B'C'$.

Proof: There is a triangle $A''B''C'' \sim A'B'C'$ with proportionality constant k . Therefore, $\angle c'' = \angle c'$, $a'' = ka'$, $c'' = kc'$. Thus $\angle c$, $\angle c''$ are right angles; $a = a''$; $c = c''$ and $\triangle ABC \cong \triangle A''B''C''$ by Hypotenuse-Leg Congruence Theorem. Thus $\triangle ABC \sim \triangle A'B'C'$.

Problem Set 7-8

1.

	AB	BC
(a)	5	$5\sqrt{3}$
(b)	$3\sqrt{2}$	$3\sqrt{2}$
(c)	$4\sqrt{3}$	4
(d)	2	$2\sqrt{3}$
(e)	$6\sqrt{2}$	6
(f)	$\frac{9\sqrt{3}}{2}$	$4\frac{1}{2}$

2. First part of proof:

Hypothesis: $\triangle ABC$, m $\angle C = 90$, $AC = BC$

Prove: $(AC, BC, AB) \underset{p}{\equiv} (1, 1, \sqrt{2})$.

Proof: Let $AC = BC = x$. Then by the Pythagorean Theorem, $(AB)^2 = x^2 + x^2$. Thus $AB = x\sqrt{2}$. Since $x, x, x\sqrt{2}$ are proportional to $(1, 1, \sqrt{2})$ with proportionality factor x , a positive quantity,

$$(AC, BC, AB) \underset{p}{\equiv} (1, 1, \sqrt{2}) .$$

Second part of proof:

Hypothesis: $\triangle ABC$ with $(AC, BC, AB) \underset{p}{\equiv} (1, 1, \sqrt{2})$.

Prove: $\triangle ABC$ has m $\angle C = 90$, $AC = BC$.

Proof: $AC = k \cdot 1$, $BC = k \cdot 1$. Therefore $AC = BC$. Since $AB = k\sqrt{2}$ and $(k \cdot 1)^2 + (k \cdot 1)^2 = (k\sqrt{2})^2$, $\triangle ABC$ must be a right triangle by the converse of the Pythagorean Theorem. Therefore, the angle opposite the longest side, $\overline{AB}$, that is, $\angle C$, is the right angle.

3. (a) $10\sqrt{2}$
(b) $5\sqrt{2}$
(c) $\frac{9}{\sqrt{2}}$ or $\frac{9}{2}\sqrt{2}$
(d) $3\sqrt{2}$
(e) 3
(f) 10

4.

	a	b	c
(a)	10	$10\sqrt{3}$	20
(b)	5	$5\sqrt{3}$	10
(c)	$3\sqrt{3}$	9	$6\sqrt{3}$
(d)	9	$9\sqrt{3}$	18
(e)	6	$6\sqrt{3}$	12
(f)	$6\sqrt{3}$	18	$12\sqrt{3}$

5. $3\sqrt{3}$

6. $4\sqrt{3}$

7. $6\sqrt{2}$

8. $3\sqrt{2}$

9. $90\sqrt{2}$ feet

10. 150 feet

11. a, b, g, h, k belong to a triangle similar to the 3, 4, 5 right triangle.

d, i, j belong to one similar to the 5, 12, 13 triangle.

e, f belong to one similar to the $1, \sqrt{3}, 2$ triangle.

l belongs to one similar to the $1, 1, \sqrt{2}$ triangle.

c does not belong to one similar to those given.

12. (a) 8

(b) 9

(c) 7

(d) 36

(e) $3\sqrt{3}$

(f) $\sqrt{3}$

(g) 6

(h) 2

(i) 2.5

(j) 10

(k) $4\sqrt{2}$

(l) $\sqrt{2}$

Answers to Review Problems

1. $a = kc$ and $b = kd$.

2. (a) 5

   (b) $\frac{1}{13}$

   (c) $(3, x, 3 + x) \underset{p}{\equiv} (39, 65, 104)$

3. 0 . Symmetric and transitive properties, for instance, would no longer hold as general properties as stated in text.

4. 1. If $(a, b, c) \underset{p}{\equiv} (p, q, r)$ then

   $(a, b, c, a + b + c) \underset{p}{\equiv} (p, q, r, p + q + r)$.

   Thus $a + b + c = k(p + q + r)$, but the sum of the angles must be 180 . Therefore, $180 = k \cdot 180$ and $k = 1$.

5. Positive

6. If the corresponding angles are congruent and corresponding sides proportional.

7. $x = 6\frac{2}{3}$. $AC = 8$. $BC = 9$.

8. (a) 10

   (b) 6

   (c) 6

   (d) 4

   (e) 12

   (f) $3\sqrt{3}$

   (g) $\frac{4}{3}\sqrt{3}$ or $\frac{4}{\sqrt{3}}$

   (h) $2\sqrt{2}$

   (i) $5\sqrt{3}$

   (j) 4

   (k) 6

   (l) $5\frac{1}{4}$

9. Yes. $\frac{k}{m}$.

10. $(b, h, x) \underset{p}{\equiv} (a, c - x, h)$

    $(a, c - x, h) \underset{p}{\equiv} (c, a, b)$

11. $a = kb$, $h = k(c - x)$.

12. $m\angle A = 60$, $m\angle B = 30$, $m\angle C = 90$.

13. $m\angle A = 45$, $m\angle B = 45$, $m\angle C = 90$.

14. (a) $AY = \sqrt{2}$, $AZ = \sqrt{3}$, $AB = \sqrt{4} = 2$.

(b) $AC = \sqrt{5}$.

(c) Continue the pattern once more to get 6 . Continuing the pattern, the points X, Y, Z, B, C, ... , are contained in a spiral, sometimes referred to as the "Square Root Spiral."

15. 13

16. (a) $AC = 7$

(b) No. $\triangle ABC \sim \triangle EBD$; however, BD, BA are not proportional to BE, BC .

17. 15, 18, 24 .

18. Let $\frac{AC}{CE} = k$, then $AC = k \cdot CE$ and likewise, $BC = k \cdot CD$. We know that k is a positive number because it represents the quotient of two positive numbers. Therefore, $(AC, BC) \underset{p}{=} (CE, CD)$. Also, $\angle ACB \cong \angle ECD$ since they are vertical angles. Thus $\triangle ACB \sim \triangle ECD$ by S.A.S. Similarity Theorem. Since corresponding angles of similar triangles are congruent, $\angle A \cong \angle E$. Therefore, since these alternate interior angles are congruent, $\overline{AB} \parallel \overline{DE}$.

19. $\angle ACB \cong \angle EDB$ since they are both right angles formed by perpendicular lines. $\angle EBD \cong \angle ABC$ by the reflexive property of congruence. Therefore, $\triangle EBD \sim \triangle ABC$ by the A.A. Similarity Theorem.

20. 12